Eigenlogic

Zeno Toffano

zeno.toffano@centralesupelec.fr

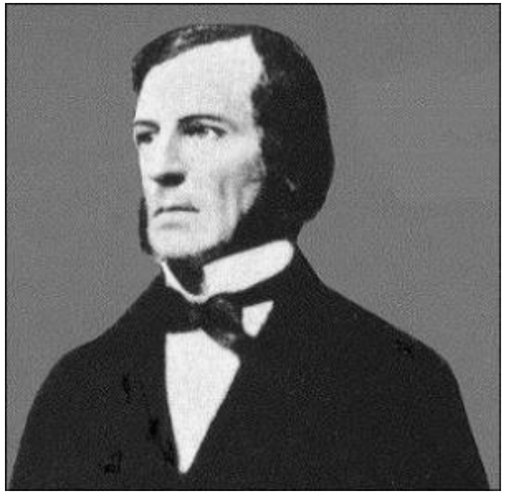
CentraleSupélec, Gif-sur-Yvette, France
Laboratoire des Signaux et Systèmes, UMR8506-CNRS
Université Paris-Saclay, France

Outline

- George Boole's method and Eigenlogic
- Binary Eigenlogic : {0,1} and {+1,-1}
- Many-valued and Fuzzy Eigenlogic
- Logic with operators and quantum physics
- The Eigenlogic program
- Bell inequalities
- Qubits and quantum circuits
- Quantum robots

George Boole's method and Eigenlogic

George Boole : truth values " 0 " and " 1 "



George Boole in 1847 [a] gave a mathematical symbolism for logical propositions.

The **conjunction** (AND) of 2 logical propositions X and Y is the product: $xy = yx$

x ("elective" symbol) acts as a selection operator on y (also y on x)

applied on itself the proposition does not change resulting in: $x^2 = x$

this equation was considered by George Boole the "**fundamental law of thought**"! [b]

the only solutions of this equation are the numbres 0 and 1 representing "False" and "True" respectively.

Same equation written as: $x(1 - x) = 0$ the logical **law of non contradiction**

showing that x is an **idempotent** symbol (projector) orthogonal to $(1 - x)$ (its **complement**)

one also has $x + (1 - x) = 1$ the logical **law of the excluded middle**

The method was extended by G. Boole in the continuous interval [0,1] to give one of the first mathematical formalizations of probabilities in [b].

[a] Boole, G. **The Mathematical Analysis of Logic. Being an Essay To a Calculus of Deductive Reasoning**, (1847)

[b] Boole, G. **An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities**, (1854)

the birth of truth tables : Pierce, Wittgenstein and Post

(excerpts from [a])

*"Truth tables were introduced by **Charles Sanders Peirce** (1839-1914) in the early 1880's [b] which attracted little attention at the time"*
*"Truth tables were rediscovered and tautologies discovered, simultaneously and independently by **Ludwig Wittgenstein** (1889-1951) [c] and by **Emil Leon Post** (1897-1954) [d] "*

Post also established the consistency and completeness of propositional calculus.
Logical truth tables (semantics $\vDash$) are axiomatic at the same level as logical connective canonical forms (syntax $\vdash$).

C.S. Peirce

E.L. Post

5.101 The truth-functions of a given number of elementary propositions can always be set out in a schema of the following kind:

$(TTTT)\,(p, q)$ Tautology (If p then p, and if q then q.) $(p \supset p \,.\, q \supset q)$
$(FTTT)\,(p, q)$ In words: Not both p and q. $(\sim(p \,.\, q))$
$(TFTT)\,(p, q)$ „ „ : If q then p. $(q \supset p)$
$(TTFT)\,(p, q)$ „ „ : If p then q. $(p \supset q)$
$(TTTF)\,(p, q)$ „ „ : p or q. $(p \vee q)$
$(FFTT)\,(p, q)$ „ „ : Not q. $(\sim q)$
$(FTFT)\,(p, q)$ „ „ : Not p. $(\sim p)$
$(FTTF)\,(p, q)$ „ „ : p or q, but not both. $(p \,.\, \sim q : \vee : q \,.\, \sim p)$
$(TFFT)\,(p, q)$ „ „ : If p then q, and if q then p. $(p \equiv q)$
$(TFTF)\,(p, q)$ „ „ : p
$(TTFF)\,(p, q)$ „ „ : q
$(FFFT)\,(p, q)$ „ „ : Neither p nor q. $(\sim p \,.\, \sim q$ or $p|q)$
$(FFTF)\,(p, q)$ „ „ : p and not q. $(p \,.\, \sim q)$
$(FTFF)\,(p, q)$ „ „ : q and not p. $(q \,.\, \sim p)$
$(TFFF)\,(p, q)$ „ „ : q and p. $(q \,.\, p)$
$(FFFF)\,(p, q)$ Contradiction (p and not p, and q and not q.) $(p \,.\, \sim p \,.\, q \,.\, \sim q)$

Wittgenstein's Tractatus

L. Wittgenstein

[a] K. Menger, **Reminiscences of the Vienna Circle and the Mathemathical Colloquium**, Springer 1994, (1942).
[b] C.S. Peirce, **On the Algebra of Logic: A Contribution to the Philosophy of Notation**, J. of Mathematics, Vol. 7 , (1885).
[c] L. Wittgenstein, **Tractatus Logico-Philosophicus**, Wien (1918), translated and published in Cambridge, (1921).
[d] E.L. Post, **Introduction to a General theory of Elementary Propositions**, American J. of Mathematics 43: 163–185, (1921).

using Boole's method for generating logical functions [*]

Boole's idempotent logical functions $f \in \{0,1\}$ are expressed in an arithmetical form (not modulo 2)

a logical function of two arguments is expressed by a **bilinear form** of the symbols x and y and the **truth values** $f(a,b)$

$$f = f(0,0)(1-x)(1-y) + f(0,1)(1-x)y + f(1,0)x(1-y) + f(1,1)xy$$

Negation is the **complementation** by subtracting f from the number 1: $\bar{f} = 1 - f$

generalizes to any number of arguments (arity)

TABLE 1. The four single argument logical elective functions

Function $f_i^{[1]}$	Operator	Truth table	Canonical form	Arithmetic form
$f_0^{[1]}$	F	0 0	0	0
$f_1^{[1]}$	$\bar{A}$	1 0	$(1-x)$	$1-x$
$f_2^{[1]}$	A	0 1	x	x
$f_3^{[1]}$	T	1 1	$(1-x)+x$	1

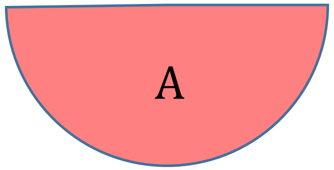
TABLE 2. The sixteen two argument logical elective functions

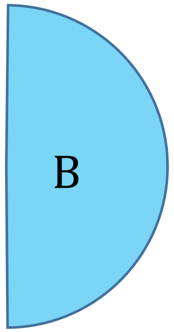
Funct. $f_i^{[2]}$	Connective for A and B	Truth table	Canonical form	Arithmetic form
$f_0^{[2]}$	F	0 0 0 0	0	0
$f_1^{[2]}$	$NOR\,,\ \bar{A} \wedge \bar{B}$	1 0 0 0	$(1-x)(1-y)$	$1-x-y+xy$
$f_2^{[2]}$	$\overline{A \Leftarrow B}$	0 1 0 0	$(1-x)y$	$y-xy$
$f_3^{[2]}$	$\bar{A}$	1 1 0 0	$(1-x)(1-y)+(1-x)y$	$1-x$
$f_4^{[2]}$	$\overline{A \Rightarrow B}$	0 0 1 0	$x(1-y)$	$x-xy$
$f_5^{[2]}$	$\bar{B}$	1 0 1 0	$(1-x)(1-y)+xy$	$1-y$
$f_6^{[2]}$	$XOR\,,\ A \oplus B$	0 1 1 0	$(1-x)y+x(1-y)$	$x+y-2xy$
$f_7^{[2]}$	$NAND\,,\ \bar{A} \vee \bar{B}$	1 1 1 0	$(1-x)(1-y)+(1-x)y+x(1-y)$	$1-xy$
$f_8^{[2]}$	$AND\,,\ A \wedge B$	0 0 0 1	xy	xy
$f_9^{[2]}$	$A \Leftrightarrow B$	1 0 0 1	$(1-x)(1-y)+xy$	$1-x-y-2xy$
$f_{10}^{[2]}$	B	0 1 0 1	$(1-x)y+xy$	y
$f_{11}^{[2]}$	$A \Rightarrow B$	1 1 0 1	$(1-x)(1-y)+(1-x)y$	$1-x+xy$
$f_{12}^{[2]}$	A	0 0 1 1	$x(1-y)+xy+xy$	x
$f_{13}^{[2]}$	$A \Leftarrow B$	1 0 1 1	$(1-x)(1-y)+x(1-y)+xy$	$1-y+xy$
$f_{14}^{[2]}$	$OR\,,\ A \vee B$	0 1 1 1	$(1-x)y+x(1-y)+xy$	$x+y+xy$
$f_{15}^{[2]}$	T	1 1 1 1	$(1-x)(1-y)+(1-x)y+x(1-y)+xy$	1

[*] Toffano, Z. **Eigenlogic in the Spirit of George Boole**. Logica Universalis, Birkhäuser-Springer, 14, 175–207 (2020).

logical forms and diagrams

Elementary propositions: A **,** B



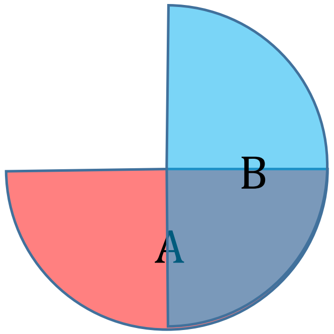


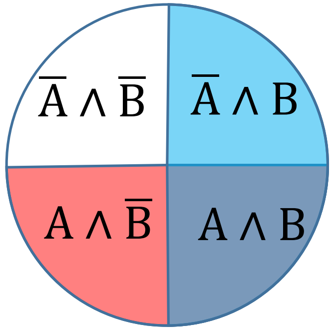
<u>**SOP** (Sum Of Products) canonical form</u>

disjunction (∨, OR) of conjunctions (∧, AND)

Conjunction, AND : $\mathrm{A} \wedge \mathrm{B}$

in arithmetical form: ab



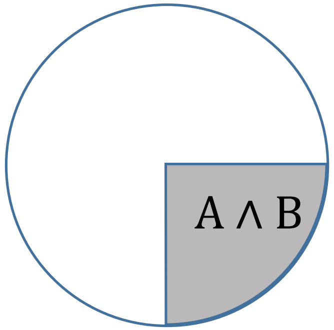


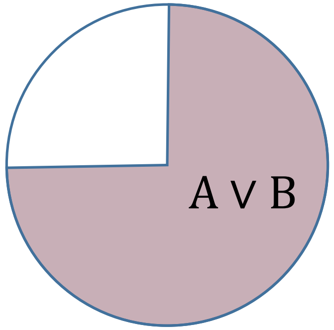
Disjunction, OR : $\mathrm{A} \vee \mathrm{B} = (\overline{\mathrm{A}} \wedge \mathrm{B}) \vee (\mathrm{A} \wedge \overline{\mathrm{B}}) \vee (\mathrm{A} \wedge \mathrm{B})$

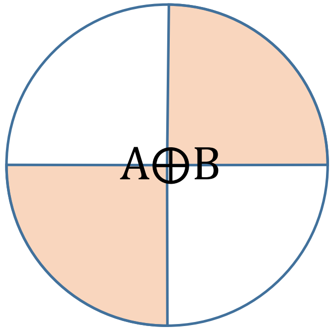
in arithmetical form: $(1-a)b + a(1-b) + ab = a + b - ab$

Exclusive disjunction, XOR : $\mathrm{A} \oplus \mathrm{B} = (\overline{\mathrm{A}} \wedge \mathrm{B}) \vee (\mathrm{A} \wedge \overline{\mathrm{B}})$

in arithmetical form: $(1-a)b + a(1-b) = a + b - 2ab$







<u>**Reed-Muller** canonical form</u>

exclusive disjunction (⊕, XOR) of conjunctions (∧, AND)

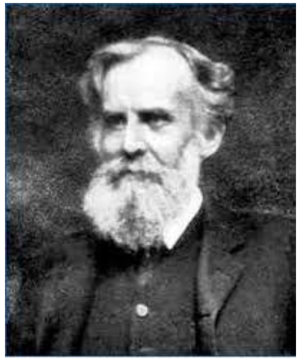
e.g. **Disjunction, OR :** $\mathrm{A} \vee \mathrm{B} = (\mathrm{A} \oplus \mathrm{B}) \vee (\mathrm{A} \wedge \mathrm{B}) = \mathrm{A} \oplus \mathrm{B} \oplus (\mathrm{A} \wedge \mathrm{B})$

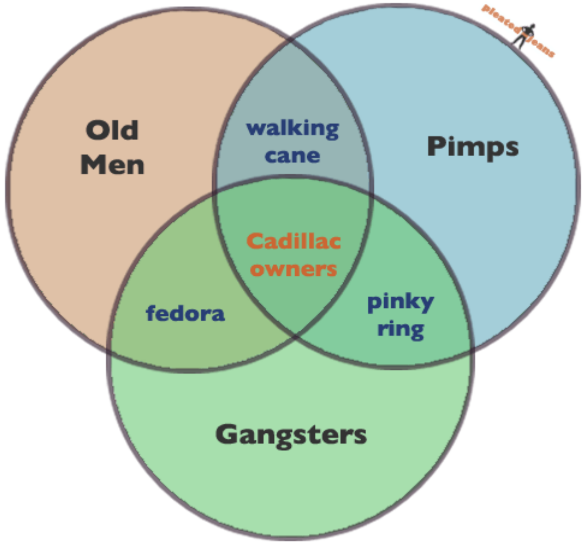
<u>**De Morgan** logical duality</u>

$$\overline{\mathrm{A}} \wedge \overline{\mathrm{B}} = \overline{\mathrm{A} \vee \mathrm{B}}$$

in arithmetical form: $(1-a)(1-b) = 1 - (a + b - ab) = 1 - a - b + ab$

Venn Diagrams



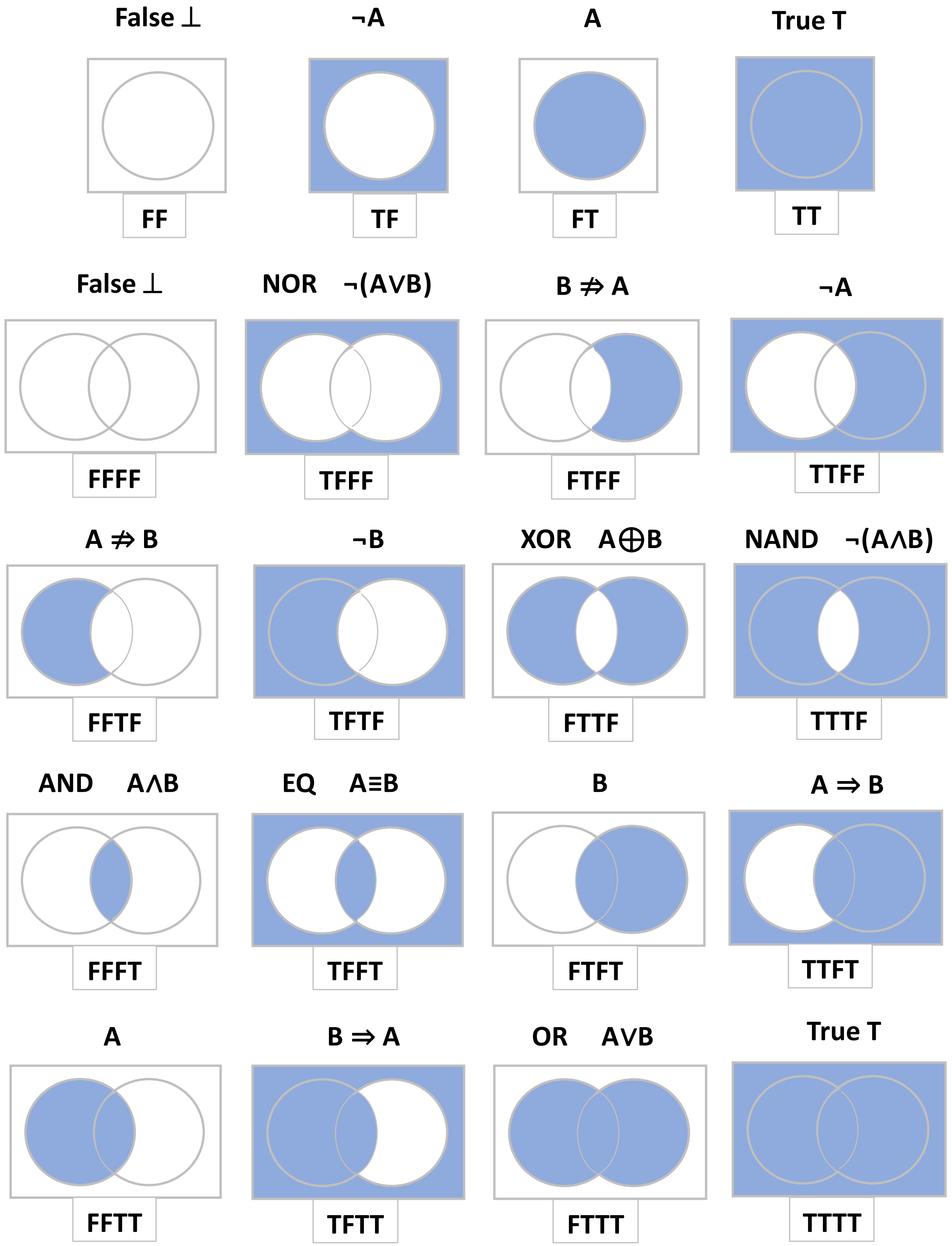


John Venn following exactly Boole’s arithmetical approach illustrated all logical connectives in his **Venn diagrams** (1881) [*]

The diagrams have a direct correspondence with **set theory** by the operations of **Intersection** ∩ and **Union** ∪ of sets (here surfaces).

Are widely used in **probability theory** and **information theory** for the illustrations of different representations (independent, relative, conditional…)

[*] Venn, J. **Symbolic Logic.**
Macmillan and Company, London UK (1881)



Eigenlogic definition

Eigenlogic: a logical method using operators in linear algebra **[a,b,c]**

logical operators ⟺ logical connectives

eigenvalues of logical operators ⟺ truth values

eigenvectors of logical operators ⟺ interpretations (propositional cases)

Eigenlogic uses the <u>Kronecker product</u> to scale-up to more logical arguments (arity).

A single <u>seed operator</u> generates the entire logic.

[a] Dubois, F., Toffano, Z., **Eigenlogic: A Quantum View for Multiple-Valued and Fuzzy Systems**,
in: de Barros J., Coecke B., Pothos E. (eds) Quantum Interaction. QI 2016. Lecture Notes in Computer Science, vol 10106. Springer. (2017)

[b] Toffano, Z., **Eigenlogic in the Spirit of George Boole**. Logica Universalis, Birkhäuser-Springer, 14, 175–207. (2020)

[c] Toffano Z, Dubois F., **Adapting Logic to Physics: The Quantum-Like Eigenlogic Program**. Entropy. ; 22(2):139. (2020)

Binary Eigenlogic : {0,1} and {+1,-1}

Eigenlogic: one-qubit Boolean logical operators

The qubits $|1\rangle$ and $|0\rangle$ define the **computational basis** (the "z" base): $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
eigenvectors of the **Pauli matrix** $\boldsymbol{\sigma}_z = \begin{pmatrix}+1 & 0\\0 & -1\end{pmatrix} = \mathrm{diag}(+1, -1)$

Choice of the **logical seed projector** $\boldsymbol{\Pi} = |1\rangle\langle 1|$ (density matrix of qubit $|1\rangle$)

Logical operators as a linear development (equivalent to Boole's method) gives the **spectral decomposition** :

$$\boldsymbol{F} = f(0)(\mathbb{I} - \boldsymbol{\Pi}) + f(1)\boldsymbol{\Pi} = \begin{pmatrix}f(0) & 0\\0 & f(1)\end{pmatrix} = \mathrm{diag}(f(0), f(1))$$

the cofactors $f(0)$ and $f(1)$ are the eigenvalues *i.e.* the truth values of the logical connective.

Negation is obtained by complementation (substracting from the identity operator) : $\overline{\boldsymbol{F}} = \mathbb{I} - \boldsymbol{F}$

other choices of logical bases are possible: *e.g.* the "x" base with the seed $\boldsymbol{\Pi}_- = |-\rangle\langle -|$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Eigenlogic: two-qubit Boolean logical operators

Making use of the **Kronecker product** $\otimes$ to scale up to more arguments (as done in quantum computing)

Scaling to 2-qubit logical operators with the 4 basis projection operators (pure quantum-state density matrices):

$$\boldsymbol{\rho}_{11} = |11\rangle\langle 11| = \boldsymbol{\Pi}\otimes\boldsymbol{\Pi} \quad ; \quad \boldsymbol{\rho}_{10} = |10\rangle\langle 10| = \boldsymbol{\Pi}\otimes(\mathbb{I}-\boldsymbol{\Pi})\ ;$$
$$\boldsymbol{\rho}_{01} = |01\rangle\langle 01| = (\mathbb{I}-\boldsymbol{\Pi})\otimes\boldsymbol{\Pi} \quad ; \quad \boldsymbol{\rho}_{00} = |00\rangle\langle 00| = (\mathbb{I}-\boldsymbol{\Pi})\otimes(\mathbb{I}-\boldsymbol{\Pi})$$

All 16 logical arity-2 operators are directly obtained by the bilinear development (G. Boole's method)

Giving the **spectral decomposition of the operator** :

$$\boldsymbol{F} = f(0,0)|00\rangle\langle 00| + f(0,1)|01\rangle\langle 01| + f(1,0)|10\rangle\langle 10| + f(1,1)|11\rangle\langle 11| =$$

$$= f(0,0)(\mathbb{I}-\boldsymbol{\Pi})\otimes(\mathbb{I}-\boldsymbol{\Pi}) + f(0,1)(\mathbb{I}-\boldsymbol{\Pi})\otimes\boldsymbol{\Pi} + f(1,0)\boldsymbol{\Pi}\otimes(\mathbb{I}-\boldsymbol{\Pi}) + f(1,1)\boldsymbol{\Pi}\otimes\boldsymbol{\Pi}$$

$$= \mathrm{diag}(f(0,0), f(0,1), f(1,0), f(1,1))$$

the truth values are $f(x,y) \in \{0,1\}$

Eigenlogic elementary propositions and logical connectives

In **propositional logic** one defines the **elementary** (or **atomic**) **propositions** P and Q in a **well-formed-formula**.

From the elementary propositions P and Q all other compound propositions can be derived.

Atomic propositions In Eigenlogic correspond to the extensions of the seed projector $\boldsymbol{\Pi}$ with the identity $\mathbb{I}$ operator:

$$\boldsymbol{P} = \boldsymbol{\Pi} \otimes \mathbb{I} = \mathrm{diag}(0,0,1,1) \quad , \quad \boldsymbol{Q} = \mathbb{I} \otimes \boldsymbol{\Pi} = \mathrm{diag}(0,1,0,1)$$

directly from $\boldsymbol{P}$ and $\boldsymbol{Q}$ all other compound logical operators are derived:

Conjunction (AND, ∧) $\qquad \boldsymbol{F}_{\mathrm{AND}} = \boldsymbol{F}_{\mathrm{P \wedge Q}} = \boldsymbol{P} \cdot \boldsymbol{Q} = \boldsymbol{\Pi} \otimes \boldsymbol{\Pi} = \mathrm{diag}(0,0,0,1)$

Disjunction (OR, ∨) $\qquad \boldsymbol{F}_{\mathrm{OR}} = \boldsymbol{F}_{\mathrm{P \vee Q}} = \boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{P} \cdot \boldsymbol{Q} = \mathrm{diag}(0,1,1,1)$

Negation is simply obtained by subtracting from the identity operator $\mathbb{I}$:

$$\boldsymbol{F}_{\mathrm{NAND}} = \mathbb{I} - \boldsymbol{F}_{\mathrm{AND}} = \mathrm{diag}(1,1,1,0) \quad ; \text{ Equivalence } \quad \boldsymbol{F}_{\Leftrightarrow} = \mathbb{I} - \boldsymbol{F}_{\mathrm{XOR}} = \mathrm{diag}(1,0,0,1)$$

Similar expressions were found by **V. Aggarwal** and **R. Calderbank** used for Quantum Error Correcting Codes in **[*]**

[*] Aggarwal, V., Calderbank, R. **Boolean Functions, Projection Operators, and Quantum Error Correcting Codes**. In: IEEE Proceedings ISIT 2007 (International Symposium Information Theory), Nice, France, pp. 2091–2095 (2007)

changing the paradigm: using values $\{+1, -1\}$ instead of $\{0, 1\}$

The **polar alphabet** $\{+1, -1\}$ has the following correspondence with the Booleans $\{0, 1\}$:

$+1$ (spin up) $\leftrightarrow$ 0 : "False" ; -1 (spin down) $\leftrightarrow$ 1 : "True"

this **binary reversible logic alphabet** is often used (implicitly) in **Ising models** and **neural networks.**

considering $x \in \{0,1\}$ one has $u \in \{+1, -1\}$ if and only if $u = 1 - 2x = (-1)^x = e^{i\pi x}$

for **operators** the equivalent form is the **Householder Transform** :

$$\boldsymbol{G} = \mathbb{I} - 2\boldsymbol{F} = (-1)^{\boldsymbol{F}} = e^{i\pi \boldsymbol{F}} = e^{i\frac{\pi}{2}} e^{-i\frac{\pi}{2}\boldsymbol{G}}$$

it is an isomorphism:

projection operators $\boldsymbol{F}$ (eigenvalues $\{0,1\}$) $\rightleftharpoons$ reversible **involution operators** $\boldsymbol{G}$ (eigenvalues $\{+1, -1\}$)

$\boldsymbol{G}$ and $\boldsymbol{F}$ have the same eigenvectors

Comparable approach in "Quantum Boolean Functions" **[*]** proposed by **A. Montanaro** and **T.J. Osborne**.

[*] Montanaro, A., Osborne, T.J., **Quantum Boolean Functions**, Chicago Journal of Theoretical Computer Science, 1, pp. 1–45 (2010)

binary operator truth tables

truth tables show
valuations: ⊨ in **semantics**

logical connectives are used for
deductions: ⊢ in **syntax**

consistency (if ⊢ A then ⊨ A)
and
completeness (if ⊨ A then ⊢ A)
are equivalent for the propositional calculus :
completeness theorem

Is not valid for **1st order logic** as shown by **Gödel's incompleteness theorem**

logical connective for P, Q	truth table {F, T}: {0, 1} or { + 1, − 1}	{0, 1} projection logical operator	{ + 1, − 1} involution logical operator
False F	F F F F	0	+ I
NOR	F F F T	I – P – Q + PQ	(1/2) (+I – U – V – UV)
P ⇍ Q	F F T F	Q – PQ	(1/2) (+I – U + V+UV)
¬P	F F T T	I – P	– U
P ⇏ Q	F T F F	P – PQ	(1/2) (+ I + U – V + UV)
¬Q	F T F T	I – Q	– V
XOR ; P⊕Q	F T T F	P + Q – 2 PQ	UV = Z⊗Z
NAND ; P↑Q	F T T T	I – PQ	(1/2) (– I – U – V + UV)
AND ; P∧Q	T F F F	PQ = Π⊗Π	(1/2) (+ I + U + V – UV)
P ≡ Q	T F F T	I – P – Q + 2 PQ	– UV
Q	T F T F	Q = I⊗Π	V = I⊗Z
P ⇒ Q	T F T T	I – P + PQ	(1/2) (– I – U + V – UV)
P	T T F F	P = Π⊗I	U = Z⊗I
P ⇐ Q	T T F T	I – Q + PQ	(1/2) (– I + U – V – UV)
OR ; P∨Q	T T T F	P + Q – PQ	(1/2) (– I + U + V + UV)
True T	T T T T	I	– I

Many-valued and Fuzzy Eigenlogic

More than binary: many-valued logic

Many-valued logic was proposed independently by **J. Łukasiewicz** [a] and **E.L. Post** [b] in 1921
born nearly simultaneously to the new mathematical theory of quantum mechanics.

With many-valued logic higher information densities can be achieved:
the information density in a m-valued system is $\log_2 m$ times larger than in a binary system

This logic has interested engineers involved in various aspects of information technology for over 40 years.

Used in HDL (Hardware Description Language) for the simulation of digital circuits and their synthesis.

Standards have been established, for example IEEE 1364MVL :

SYMBOL	MEANING
0	Logic zero
1	Logic one
Z	High-impedance state
X	Unknown logic value

The total number of logical connectives for a system of m values and n arguments is m^{m^n}.
with 2 values (binary) for arity-1 : $2^{2^1} = 4$ and for arity-2 : $2^{2^2} = 16$
with 3 values (ternary) for arity-1 : $3^{3^1} = 27$ and for arity-2 : $3^{3^2} = 19683$
The number of connectives has a doubly exponential increase with the number of values m and arity-n

[a] Jan Łukasiewicz, **On three-valued logic**, Selected Works, North-Holland, (1970), pp. 87–88 (1921)
[b] Emil Post, **Introduction to a General theory of Elementary Propositions**, American Journal of Mathematics 43: 163–185 (1921)

Cayley-Hamilton theorem and many-valued Eigenlogic

The Eigenlogic seed operator $\boldsymbol{\Lambda}$ can be any operator with m non-degenerate eigenvalues λ_i,

using **Lagrange matrix interpolation** the projector of each eigenstate is given by:

$$|\lambda_i >< \lambda_i| = \boldsymbol{\Pi}_{\lambda_i}(\boldsymbol{\Lambda}) = \prod_{j=1, j\neq i}^{m} \frac{\boldsymbol{\Lambda} - \lambda_j \mathbb{I}}{\lambda_i - \lambda_j}$$

is a polynomial in $\boldsymbol{\Lambda}$ up to the power $m - 1$ and is represented by a $m \times m$ square matrix.

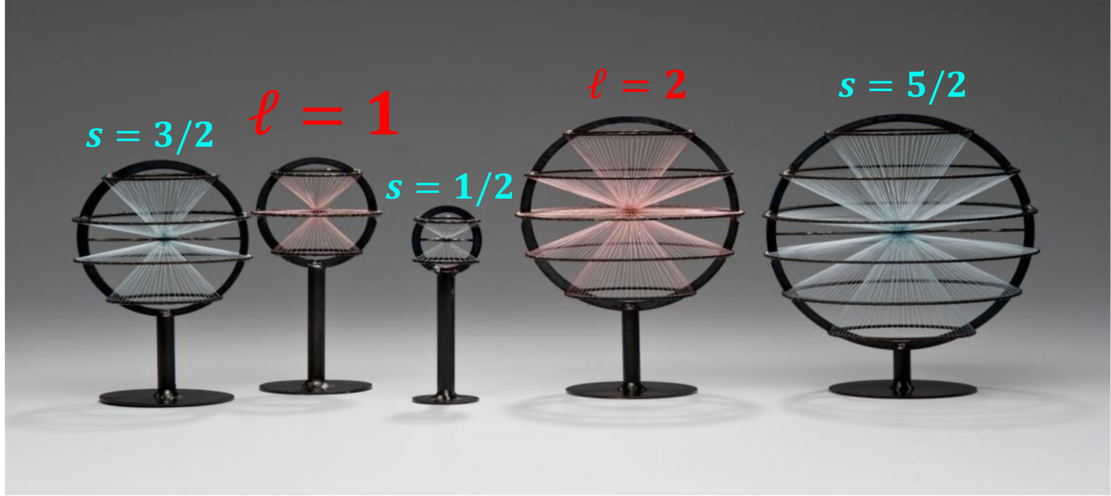
The **Cayley–Hamilton theorem** says that any finite matrix is the solution of its own characteristic equation showing that the above development is unique.

A logical operator for arity-1 is then given by the spectral decomposition with truth-values $f(\lambda_j) \in \{\ldots \lambda_i \ldots\}$:

$$\boldsymbol{F}_{_L} = \sum_{j=1}^{m} f(\lambda_j) \boldsymbol{\Pi}_{\lambda_j}(\boldsymbol{\Lambda})$$

Scaling to higher arity is obtained by extending the seed operator $\boldsymbol{\Lambda}$ with the identity.

logic of angular momentum



Spin Family (Bosons ℓ and Fermions s)
(© Julian Voss-Andreae. Photo: Dan Kvitka.)

Logical observables can be identified with
Quantum Angular Momentum

Balanced ternary logic equivalent to **Orbital Angular Momentum** (**OAM**) with $\ell = 1$.
The z component of the orbital angular momentum operator :

$$\boldsymbol{L_z} = \hbar \boldsymbol{\Lambda} = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = diag(+1,0,-1)$$

the three rank-1 projectors obtained by interpolation :

$$\boldsymbol{\Pi}_{+1} = \tfrac{1}{2}\boldsymbol{\Lambda}(\boldsymbol{\Lambda} + \mathbb{I}) \quad , \quad \boldsymbol{\Pi}_0 = \mathbb{I} - \boldsymbol{\Lambda}^2 \quad , \quad \boldsymbol{\Pi}_{-1} = \tfrac{1}{2}\boldsymbol{\Lambda}(\boldsymbol{\Lambda} - \mathbb{I})$$

For arity-2 $\boldsymbol{U}$ and $\boldsymbol{V}$ are then defined as usual in Eigenlogic :

$$\boldsymbol{U} = \boldsymbol{\Lambda} \otimes \mathbb{I} \quad , \quad \boldsymbol{V} = \mathbb{I} \otimes \boldsymbol{\Lambda}$$

In many-valued logic the Min and Max are the equivalent of AND and OR :

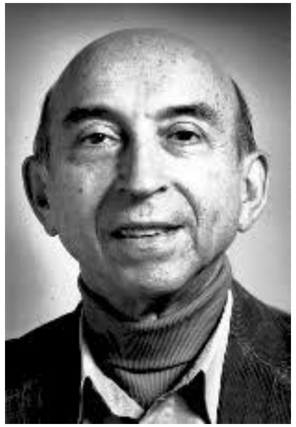
$$\boldsymbol{Min}(\boldsymbol{U}, \boldsymbol{V}) = \frac{1}{2}\left(\boldsymbol{U} + \boldsymbol{V} + \boldsymbol{U}^2 + \boldsymbol{V}^2 - \boldsymbol{U}\cdot\boldsymbol{V} - \boldsymbol{U}^2\cdot\boldsymbol{V}^2\right) = diag(1,1,1,1,0,0,\ 1,0,-1)$$

$$\boldsymbol{Max}(\boldsymbol{U}, \boldsymbol{V}) = \frac{1}{2}\left(\boldsymbol{U} + \boldsymbol{V} - \boldsymbol{U}^2 - \boldsymbol{V}^2 + \boldsymbol{U}\cdot\boldsymbol{V} + \boldsymbol{U}^2\cdot\boldsymbol{V}^2\right) = diag(1,0,-1,0,0,-1,-1,-1,-1)$$

Min U \\ V	F	N	T
False ≡ +1	+1	+1	+1
Neutral ≡ 0	+1	0	0
True ≡ − 1	+1	0	− 1

Max U \\ V	F	N	T
False ≡ +1	+1	0	− 1
Neutral ≡ 0	0	0	− 1
True ≡ − 1	− 1	− 1	− 1

fuzzy Eigenlogic: when the logical input is not an eigenstate



In 1965 **Lotfi Zadeh** [a] proposed **fuzzy logic** to describe partial truths, truth values can take values between 0 and 1.
Fuzzy logic is grounded on the theory of **fuzzy sets**.
The relation between the theory of fuzzy sets and the probability theory has been debated for a long time.

The quantum principle of superposition of states finds a counterpart in the degree of membership to fuzzy sets: the mean value of an Eigenlogic projection operator $\boldsymbol{F}$ gives a **fuzzy measure** when the quantum state $|\psi\rangle$ is not an eigenstate of $\boldsymbol{F}$. Whereas a **crisp measure** 0 or 1 is obtained only for the eigenstates of $\boldsymbol{F}$.

The Eigenlogic **fuzzy membership function** is: $\mu = \langle\psi|\boldsymbol{F}|\psi\rangle$ with $0 \le \mu \le 1$

Fuzziness can be related to the probabilistic nature of quantum measurements (**Born rule**).
For a projective observable $\boldsymbol{P}$ measured on a quantum state $|\psi\rangle$ we have the probability (**Gleason's theorem** [b]):

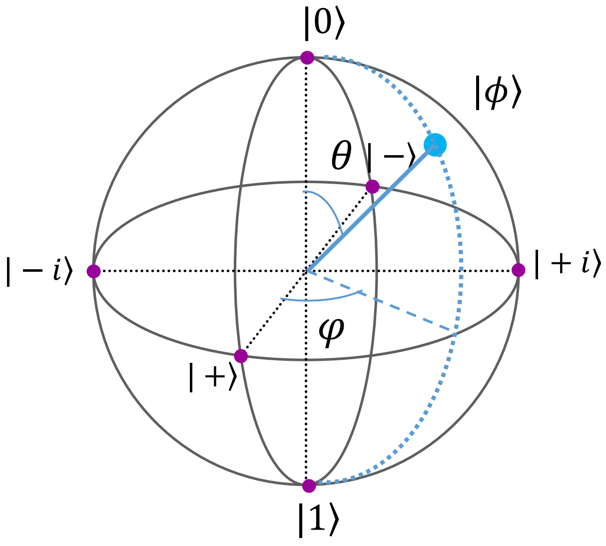
$$p_{|\psi\rangle} = \langle\psi|\boldsymbol{P}|\psi\rangle = Tr(\boldsymbol{\rho}\cdot\boldsymbol{P}) \qquad \text{with} \qquad \boldsymbol{\rho} = |\psi\rangle\langle\psi| \quad \text{the “density matrix”.}$$

A projective observable corresponds to a logical projection operator in Eigenlogic.

[a] Zadeh, L.A.: **Fuzzy sets**. Information and Control, 8 (3), 338-353, (1965)
[b] A. M. Gleason, **Measures on the closed subspaces of a Hilbert space**. Indiana U. Mathematics Journal, 6, 885–893, (1957)

Eigenlogic fuzzy conjunction, disjunction and material implication

A generic qubit state on the Bloch sphere: $|\phi\rangle = \sin\frac{\theta}{2}|0\rangle + e^{i\varphi}\cos\frac{\theta}{2}|1\rangle$,



Bloch sphere in Hilbert space

The quantum average (Born rule) of the logical projector is: $\mu(\mathrm{A}) = \langle\phi|\boldsymbol{\Pi}|\phi\rangle = \cos^2\frac{\theta}{2}$
and the complement: $\mu(\bar{\mathrm{A}}) = \langle\phi|(\mathbb{I} - \boldsymbol{\Pi})|\phi\rangle = \sin^2\frac{\theta}{2} = 1 - \cos^2\frac{\theta}{2} = 1 - \mu(\mathrm{A})$

which satisfies the condition of fuzzy logic for the complement (negation) of a fuzzy set.

The fuzzy membership function for $\boldsymbol{P}$ and $\boldsymbol{Q}$ by performing the quantum average on :

$|\psi\rangle = |\phi_p\rangle \otimes |\phi_q\rangle$ with $p = \left(\cos\frac{\theta_p}{2}\right)^2$ and $q = \left(\cos\frac{\theta_q}{2}\right)^2$

$\mu(\mathrm{P}) = \langle\psi|\boldsymbol{P}|\psi\rangle = p(1 - q) + p \cdot q = p$; $\mu(\mathrm{Q}) = \langle\psi|\boldsymbol{Q}|\psi\rangle = q$

The fuzzy Eigenlogic measure of logical operators are :

Conjunction (AND) $\mu(\mathrm{P} \wedge \mathrm{Q}) = \langle\psi|\boldsymbol{P} \cdot \boldsymbol{Q}|\psi\rangle = \langle\psi|\boldsymbol{\Pi} \otimes \boldsymbol{\Pi}|\psi\rangle = p \cdot q = \mu(\mathrm{P}) \cdot \mu(\mathrm{Q})$

Disjunction (OR) $\mu(\mathrm{P} \vee \mathrm{Q}) = p + q - p \cdot q = \mu(\mathrm{P}) + \mu(\mathrm{Q}) - \mu(\mathrm{P}) \cdot \mu(\mathrm{Q})$

Material Implication $\mu(\mathrm{P} \Rightarrow \mathrm{Q}) = 1 - p + p \cdot q = 1 - \mu(\mathrm{P}) + \mu(\mathrm{P}) \cdot \mu(\mathrm{Q})$

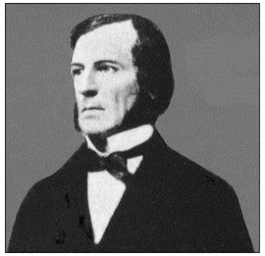
In the language of fuzzy logic these are **Product t-norms** (triangular norm as a product for the fuzzy conjunction)

The disjunction $\mu(\mathrm{P} \vee \mathrm{Q})$ corresponds to the **inclusion-exclusion** expression for probabilities due to **H. Poincaré** **[*]**

[*] Poincaré, H. **Calcul des Probabilités**; Gauthier-Villars: Paris, France, 1912

Logic with operators and quantum physics

operators in Logic



In 1847 **G. Boole** uses symbols (elective) that act as idempotent operators
<u>less known</u>: in 1848 in [a], he gave a logical interpretation of unitary **quaternions**.

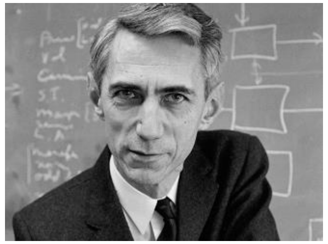
C. S. Pierce used matrices to build his logical formalism at the end of the XIXth century.



In 1921 **L. Wittgenstein** states in the Tractatus [b] that all propositions can be derived by repeated application of the operator N to the elementary propositions.

In 1924, **M. Schönfinkel** [c] introduced an operator based method in logic.
H. Curry named it successively **Combinatory Logic** and improved the method in [d] which lead successively to **lambda calculus** introduced **A. Church**.

In 1943 **E.L. Post** [e] created a string-manipulation system (**Post production system**) and proved the remarkable **Normal-form Theorem** $g\$ \to \$h$



<u>Boolean logical functions</u> as used nowadays in **digital circuits** were introduced by **C.E. Shannon** in his Master's thesis in 1938 [f] to represent logical binary switching functions.

[a] Boole, G. **Notes on quaternions**. Philos. Mag, 33, 278–280, (1848)
[b] Wittgenstein, L. **Tractatus Logico-Philosophicus**, prop. 6.001, Routledge (1921)
[c] Schönfinkel, M. **Über die Bausteine der mathematischen Logik**. Math. Ann, 92, 305–316. (1924)
[d] Curry, H.B.; Feys, R. **Combinatory Logic**; North-Holland Co: Amsterdam, The Netherlands, (1958)
[e] Post, E. **Formal Reductions of the General Combinatorial Decision Problem**, American Journal of Mathematics 65 (2), 197-215. (1943)
[f] Shannon, C.E. **A symbolic analysis of relay and switching circuits**. Trans. AIEE, 57(12), 713–723 (1938)

why adapting logic to physics ?

Logic: greek word **logos**, means both **speech** and **reason**,

Logic defines languages whose **syntax** constructs formal languages and whose **semantics** interprets them.

Syntax should have a link with the structure of Physics (symmetry, geometry, particles, waves…)

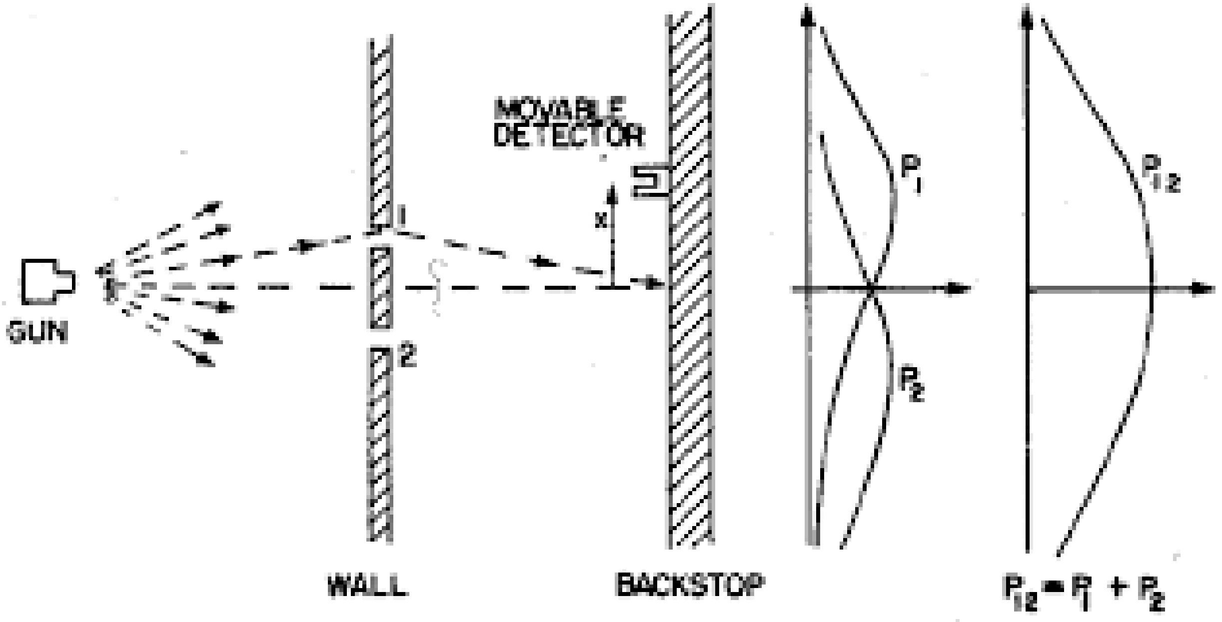
Semantics should correspond to measurement in Physics (values: discrete, rational, complex, random…)

Looking for more “physical” representations of logic. How ?

Different possibilities:

- Exploiting Boolean logic {0,1} by **operators** (quantum logic, quantum computing, …)
- Using alternative binary logical values, **negative numbers** e.g. $\{-1, +1\}$: spin, Ising model, neural networks…
- Using **many-valued** logical values: Qudits, Fourier Transform,…
- Using **fuzzy** logic: continuous values, quantum probabilistic interpretation…
- Representing the logic by operators instead of functions to handle with **noncommutativity** and **reversibility**

non-distributive "quantum propositions"



from: "The Feynman Lectures on Physics", vol. III, p. 1-2, Addison Wesley (1965)

Difficulty of explaining quantum experiments by the means of propositional logic

For the "double slit experiment" three propositions:

prop. A : "the electron is detected at the point x"

prop. B: "the electron went through slit 1"

prop. C: "the electron went through slit 2"

This experimental configuration does not verify the distributive property of classical propositional logic:

$$A \wedge (B \vee C) \not\equiv (A \wedge B) \vee (A \wedge C)$$

this has motivated the research on more "adapted" logical systems.

quantum logic : projections as propositions

M. H. Stone gave the conditions for operations on projectors and commutativity **[a]** and established that each binary logical proposition corresponds by duality to the set of all its true valuations (**Stone Duality**).

M.H. Stone

J. von Neumann considered measurement projection operators as propositions in 1932 **[b]**
and also stated that a quantum state $|\psi\rangle$ can be represented by a **density matrix** (rank-1 projection operator) :

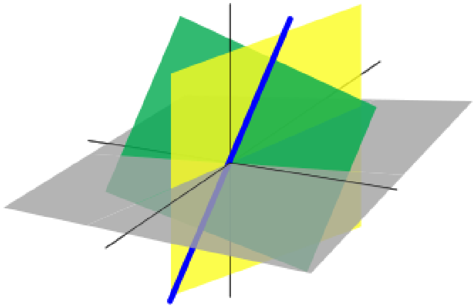
$$\boldsymbol{\rho} = |\psi\rangle\langle\psi|$$

G. Birkhoff

J. Von Neumann

Quantum Logic proposed by **G. Birkhoff** and **J. von Neumann** in 1936 **[c]** suggested the replacement of Boolean algebras with the lattice of closed subspaces of a (finite) Hilbert space.

based on projection measurements it is a **non-Boolean** logic and fails to meet distributive properties as expected by quantum mechanics.
<u>Problem: no satisfying way to do implication</u>



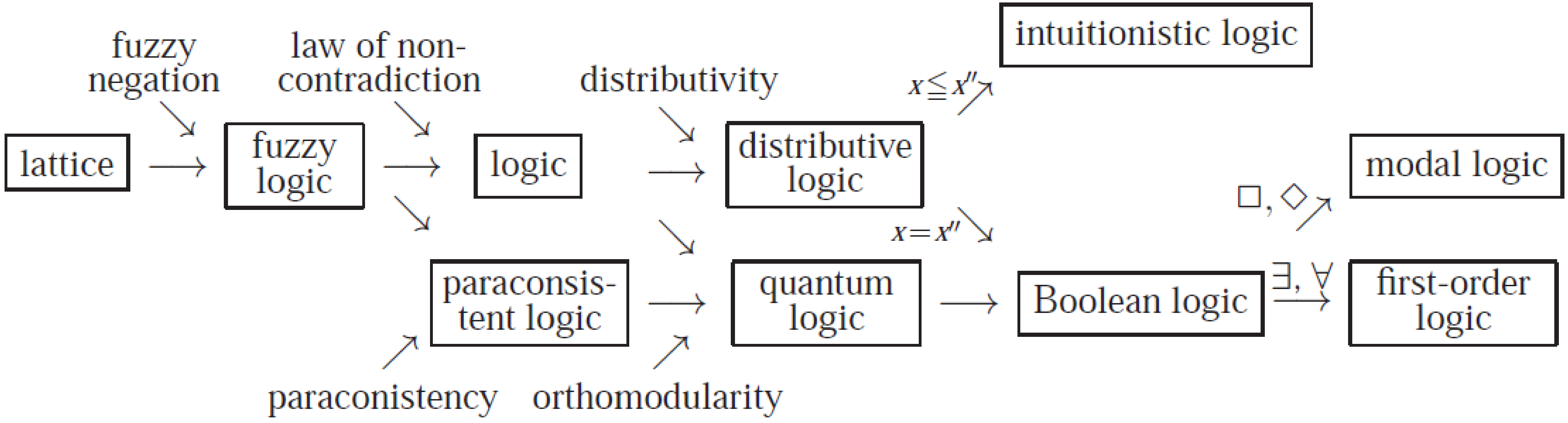
[a] Marshall Harvey Stone: **Linear Transformations in Hilbert Space and Their Applications to Analysis**, p.70: “Projections”, (1932)

[b] John von Neumann. **Mathematical Foundations of Quantum Mechanics**, Eng. Transl. (1955), p.249: “Projectors as Propositions”, (1932)

[c] Garret Birkhoff, John von Neumann, **The Logic of Quantum Mechanics**. The Annals of Mathematics, 2nd Ser., 37 (4), 823-843 (1936)

The Eigenlogic program

trying to classify Eigenlogic



Following the classification of the algebraic hierarchy of logics proposed **Andreas de Vries** [*].

Eigenlogic encompassing Boolean, many-valued, fuzzy, quantum and first order could fit into this diagram

[*] A. de Vries, **Algebraic hierarchy of logics unifying fuzzy logic and quantum logic**, arXiv:0707.2161 (2007)
and in **Quantum Computation**, chap 13 **Quantum Logic** Ed. Books on Demand (2012)

debate around traditional quantum logic

Bitrkhoff and von Neumann quantum logic was much debated with many promoters but also detractors.

It is still not considered an "operational tool" for quantum computing

An alternative interpretation named **geometry of interaction**, in contrast to quantum logic, was proposed by the French logician **Jean-Yves Girard** stating in **[a]**:

"The idea would be to revisit logic in relation with this phenomenon ignored, despised, by logicians – who treated it with contempt through their calamitous quantum logic – quantum physics. To imagine foundations, if not «quantum», at least in a quantum spirit: proportionately speaking, something of the sort Alain Connes is doing with non-commutative geometry. That is the project of the day, enough to be kept busy for a while! Which topsy-turvies the usual relation logic/quantum: instead of interpreting quantum in logic, one tries the opposite."

A development of quantum logic has been proposed recently, inspired by quantum computing research, leading to the quantum computational logic approach **[b]** by **Maria Luisa Dalla Chiara** et al. where any language formula in logic can be considered as a compact logical description of a quantum circuit.

[a] Jen-Yves Girard, **The Blind Spot: Lectures on Logic,** p. 11, European Mathematical Society (2011)

[b] M.L. Dalla Chiara, R. Giuntini, R. Leporini, G. Sergioli, **Quantum Computation and Logic—How Quantum Computers Have Inspired Logical Investigations, Trends in Logic** (Book 48); Springer: Berlin/Heidelberg, Germany (2018)

Eigenlogic is different from quantum logic

In Eigenlogic the **extension** of the seed operator $\boldsymbol{\Pi}$ with the identity operator $\mathbb{I}$
(as is done in quantum mechanics for example for the composition of two spins ½ operators)

ensures the **independence** of the **elementary propositions** represented by operators $\boldsymbol{P}$ and $\boldsymbol{Q}$

$$\boldsymbol{P} = \boldsymbol{\Pi} \otimes \mathbb{I} \quad ; \quad \boldsymbol{Q} = \mathbb{I} \otimes \boldsymbol{\Pi}$$

this is a major **<u>difference with traditional quantum logic</u>** !

In Quantum Logic **atomic propositions** are represented by **pure state density matrices** (rank-1 projection operators).
In Eigenlogic the atomic propositions $\boldsymbol{P}$ and $\boldsymbol{Q}$ <u>are not rank-1 projection operators</u>

For example considering the density matrix of the quantum state $|11\rangle$: $\boldsymbol{\rho}_{11} = |11\rangle\langle 11| = \boldsymbol{\Pi} \otimes \boldsymbol{\Pi}$

The operator corresponds in Eigenlogic to a **conjunction** (compound proposition)

$$\boldsymbol{F}_{\text{AND}} = \boldsymbol{P} \cdot \boldsymbol{Q} = \boldsymbol{\Pi} \otimes \boldsymbol{\Pi} = |11\rangle\langle 11| = \boldsymbol{\rho}_{11}$$

<u>conjunction is not a logical elementary proposition</u>

Eigenlogic syntax-semantic duality and non-commutativity

For the logical alphabet $\{+1, -1\}$

considering the 2 eigenstates $|\pm z\rangle$ of the Pauli operator $\boldsymbol{\sigma}_z$ with eigenvalues ± 1

and using he **anti-commutativity** of the Pauli operators: $\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_z = -\boldsymbol{\sigma}_z \cdot \boldsymbol{\sigma}_x$

$\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_z|\pm z\rangle = (\pm 1)\boldsymbol{\sigma}_x|\pm z\rangle = -\boldsymbol{\sigma}_z \cdot \boldsymbol{\sigma}_x|\pm z\rangle$ gives $\boldsymbol{\sigma}_z(\boldsymbol{\sigma}_x|\pm z\rangle) = (\mp 1)(\boldsymbol{\sigma}_x|\pm z\rangle)$

so $(\boldsymbol{\sigma}_x|\pm z >)$ are eigenstates of $\boldsymbol{\sigma}_z$ with eigenvalues ∓ 1 and correspond to the eigenstates $|\mp z\rangle)$

Identifying $|+z\rangle$ with qubit $|0\rangle$ and $|-z\rangle$ with qubit $|1\rangle$ gives : $\boldsymbol{\sigma}_x|0\rangle = |1\rangle$ and $\boldsymbol{\sigma}_x|1\rangle = |0\rangle$

this operation corresponds to **logical binary negation**.

So for these operators the basic logical operation of binary negation is a consequence of anti-commutativity

In this very simple example using the Pauli matrices as Eigenlogic operators, one has simultaneously:

- a **semantic representation** by the <u>eigenstructure</u> (eigenvalues ± 1 and eigenvectors $|\pm z >$) of Pauli matrix $\boldsymbol{\sigma}_z$
- a **syntactic representation** by a <u>permutation operation</u> represented by Pauli matrix $\boldsymbol{\sigma}_x$.

syntax and semantics for many-valued operators (qudits)

The quantum state logical complementation can be generalized for a d-dimensional multi-level system (**qudit**)

using the **generalized Pauli operators** given by the **Weyl-Heisenberg pairs** $\boldsymbol{X}_d$ and $\boldsymbol{Z}_d$

$$\boldsymbol{Z}_d|j\rangle = \omega_d^j|j\rangle \quad \text{with} \quad \omega_d = e^{i\frac{2\pi}{d}} \quad ; \quad \boldsymbol{X}_d|j\rangle = |j+1\rangle \quad \text{with} \quad {\boldsymbol{Z}_d}^d = {\boldsymbol{X}_d}^d = \mathbb{I}_d$$

$\boldsymbol{X}_d$ and $\boldsymbol{Z}_d$ possess the same eigenvalues and verify : $\boldsymbol{Z}_d \cdot \boldsymbol{X}_d = \omega_d\, \boldsymbol{X}_d \cdot \boldsymbol{Z}_d$

the action of the **shift operator** $\boldsymbol{X}_d$ on the state $|j\rangle$, which is an eigenstate of $\boldsymbol{Z}_d$, gives the state $|j+1\rangle$
so by applying successively this operator one can generate all the other states of the basis.

<u>The semantics is here represented by the eigenstructue of</u> $\boldsymbol{Z}_d$, the eigenvalues ω_d are the d^{th} roots of unity

<u>The syntax is represented by</u> $\boldsymbol{X}_d$ corresponding to a **many-valued negation** as expressed by **E.L. Post** in 1921.

The transformation from $\boldsymbol{Z}_d$ to $\boldsymbol{X}_d$ is the **Discrete Fourier Transform** operator $\boldsymbol{QFT}_d$ (**Quantum Fourier Transform**)

$$(\boldsymbol{QFT}_d)_{ij} = \frac{1}{\sqrt{d}}{\omega_d}^{ij} \quad ; \quad {\boldsymbol{QFT}_d}^4 = \mathbb{I}_d \quad ; \quad {\boldsymbol{QFT}_d}^{-1} \cdot \boldsymbol{Z}_d \cdot \boldsymbol{QFT}_d = \boldsymbol{X}_d$$

<u>The Quantum Fourier Transform can be seen as a mediator between logical syntax and logical semantics.</u>

towards first-order Eigenlogic

Using two maximally incompatible logical families with logical eigensystems associated to the $\boldsymbol{X}$ and $\boldsymbol{Z}$ gates (*resp.* the $\boldsymbol{\sigma}_x$ and $\boldsymbol{\sigma}_z$ Pauli operators) one gets an interesting outlook:

the quantum Grover amplification gate used in the *Grover algorithm*, corresponds to the multi-qubit involution Eigenlogic negated disjunction operator NOR in the $\boldsymbol{X}$ system.

This operator can be interpreted in the $\boldsymbol{Z}$ system as a predicative logical *existential connective* ∃.

In the language of first order logic with a 3-qubit phase oracle the Grover circuit operates the following logical proposition:

$$\exists a\ P(a)\ \equiv\ \neg(P_X \vee Q_X \vee R_X)\ [P_Z \wedge Q_Z \wedge R_Z]$$

A justification can be found in **Herbrand's fundamental theorem [*]** that provides a constructive characterization of derivability in first-order predicate logic by means of propositional (sentential) logic.

.

[*] J. Herbrand, **Recherches sur la théorie de la démonstration**, Thèses présentées à la faculté des sciences de Paris, Paris, (1930).

quantum-like Combinatory Logic

Moses Schönfinkel [a] introduced a method in logic named *Combinatory Logic*, this was part of the Hilbert program aimed to formulate all the fields of mathematics in a consistent logic system.

Haskell Curry successively improved and completed the research [b]. This led to the development of *functional programming languages* such as *Haskell*, and *Erlang*.

Combinatory logic uses abstract operators (**combinators**) to compose and to transform operators and arguments.

It permits to translate first order logic into expressions without variables using only combinators without the need of the universal quantifier ∀ and the existential quantifier ∃.

Alessandra di Pierro, in [c] considers that "…*reversible combinatory logic can in principle be used for a … translation of classical into quantum computation.*"

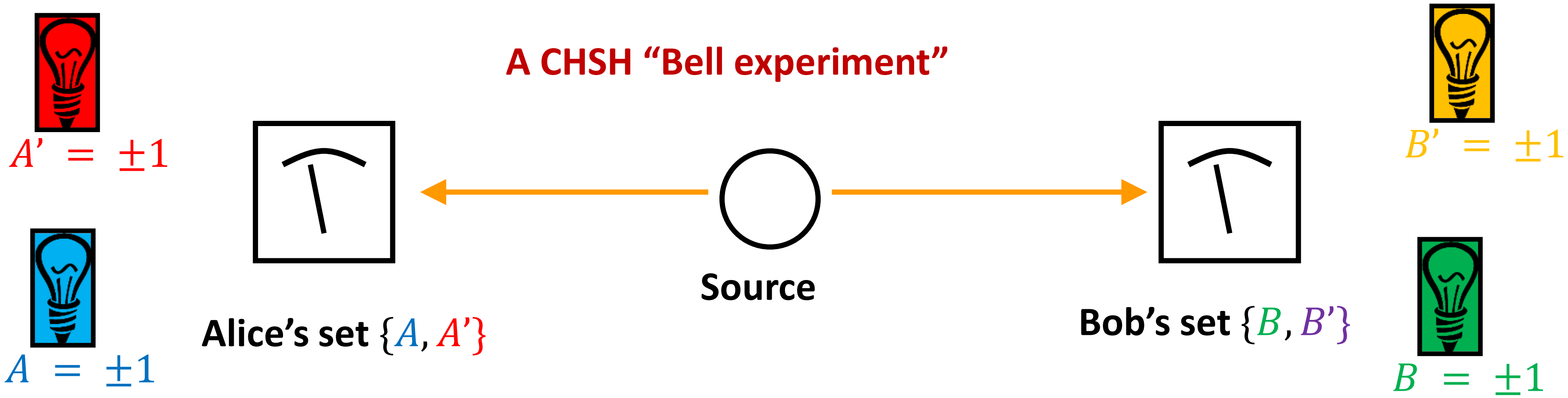
A tentative approach could consist in identifying the different operations of substitution, elimination, permutation, etc., with equivalent operations obtained using quantum gates.

tricks in q. computation could be used: $\boldsymbol{C_Z} \cdot (\boldsymbol{X} \otimes \boldsymbol{Z}) \cdot \boldsymbol{C_Z} = \boldsymbol{X} \otimes \mathbb{I}_2$ or $\boldsymbol{U}_{\mathrm{swap}} \cdot (\boldsymbol{P} \otimes \boldsymbol{Q}) \cdot \boldsymbol{U}_{\mathrm{swap}} = \boldsymbol{Q} \otimes \boldsymbol{P}$

[a] Schönfinkel, M. **Über die Bausteine der mathematischen Logik**. Math. Ann, 92, 305–316. (1924)
[b] H.B. Curry, **Combinatory Logic**, North-Holland Co: Amsterdam, (1958)
[c] A. Di Pierro, **On Reversible Combinatory Logic**, Electron. Notes Theor. Comput. Sci. 135, 25–35 (2006)

CHSH - Bell inequalities

correlations of 2 quantum particles: CHSH Bell-inequality [*]



[*] J.F. Clauser; M.A. Horne; A. Shimony; R.A. Holt, **Proposed experiment to test local hidden-variable theories**, Phys. Rev. Lett., 23 (15): 880–4, (1969)

Local properties ($A = \pm 1$, $A' = \pm 1$, $B = \pm 1$, $B' = \pm 1$)

in all 16 cases (deterministic) the expression: $AB + AB' + A'B - A'B' = A(B + B') + A'(B - B') = \pm 2$

CHSH-Bell inequality requires 16 measurements giving (average on a great number of measurements):

$$|S| = |< AB + AB' + A'B - A'B' >| \leq 2$$

But Quantum Mechanics allows: $2 < S \leq 2\sqrt{2} = 2.83$

so violates the CHSH Inequality > 2 !

The CHSH Bell inequality and probabilities

The CHSH Bell inequality is expressed with the Pauli spin operators $\boldsymbol{\sigma}_i$ along the 4 measurement directions

The CHSH measurement operator is then : $\boldsymbol{CHSH} = \boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_B + \boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_{B'} + \boldsymbol{\sigma}_{A'} \otimes \boldsymbol{\sigma}_B - \boldsymbol{\sigma}_{A'} \otimes \boldsymbol{\sigma}_{B'}$

Considering the projection operators, each term transforms as: $\boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_B = (\mathbb{I} - 2\boldsymbol{\Pi}_{\boldsymbol{A}}) \otimes (\mathbb{I} - 2\boldsymbol{\Pi}_{\boldsymbol{B}})$
replacing and simplifying :

$$\boldsymbol{CHSH} = 2\mathbb{I} - 4\boldsymbol{\Pi}_{\boldsymbol{A}} \otimes \mathbb{I} - 4\mathbb{I} \otimes \boldsymbol{\Pi}_{\boldsymbol{B}} + 4\boldsymbol{\Pi}_{\boldsymbol{A}} \otimes \boldsymbol{\Pi}_{\boldsymbol{B}} + 4\boldsymbol{\Pi}_{\boldsymbol{A}} \otimes \boldsymbol{\Pi}_{\boldsymbol{B'}} + 4\boldsymbol{\Pi}_{\boldsymbol{A'}} \otimes \boldsymbol{\Pi}_{\boldsymbol{B}} - 4\boldsymbol{\Pi}_{\boldsymbol{A'}} \otimes \boldsymbol{\Pi}_{\boldsymbol{B'}}$$

in this expression one recognizes the Eigenlogic projection and conjunction operators

To evaluate the Bell inequality parameter S one averages this operator: $S = \langle \psi | \boldsymbol{CHSH} | \psi \rangle$

By averaging the operator $\boldsymbol{\mathcal{F}} = \frac{1}{4} \boldsymbol{CHSH} - \frac{\mathbb{I}}{2}$ one obtains the **Fine inequality** for probabilities:

$$\mathcal{F} = \langle \psi | \boldsymbol{\mathcal{F}} | \psi \rangle = P(\mathrm{A} \wedge \mathrm{B}) + P(\mathrm{A} \wedge \mathrm{B}') + P(\mathrm{A}' \wedge \mathrm{B}) - P(\mathrm{A}' \wedge \mathrm{B}') - P(\mathrm{A}) - P(\mathrm{A}) = \frac{1}{4}(S - 2)$$

classically $-1 \leq \mathcal{F} \leq 0$ equivalent to $-2 \leq S \leq +2$ for entangled states one has violation of these inequalities.

George Boole already discussed these probability inequalities in 1854 as stated by **Itamar Pitowsky** in **[*]**

[*] Pitowsky I. **From George Boole To John Bell — The Origins of Bell's Inequality**. In: Kafatos M. (eds) Bell's Theorem, Quantum Theory and Conceptions of the Universe. Fundamental Theories of Physics, vol 37. Springer (1989)

the Bell CHSH inequality cases

Classical, local, separable

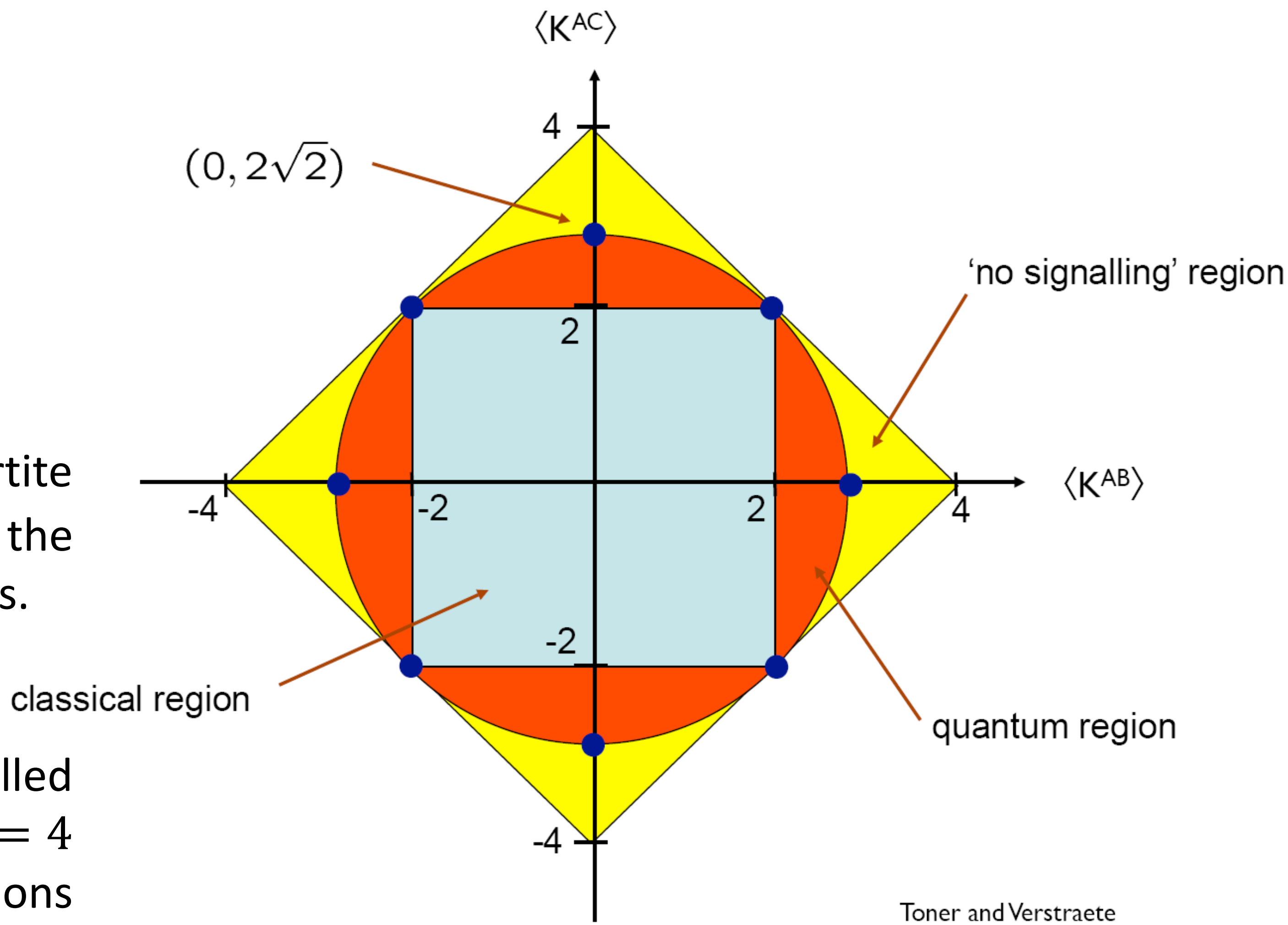
The Bell parameter S_{Bell} lies between 0 and 2. Measurements are local: $E(X,Y) = E(X)E(Y)$.

Quantum

The case $2 \leq S_{Bell} \leq 2\sqrt{2}$ achieved with bipartite quantum entangled states. $S_{Bell} = 2\sqrt{2}$ is called the Tsirelson's bound and is a limit for Quantum systems.

Post-quantum

The case between $2\sqrt{2}$ and 4 comprises the so-called "no-signalling" region. The maximum value $S_{Bell} = 4$ can be attained with logical probabilistic constructions often named **non-local PR boxes**.



Toner and Verstraete

the strange propositions of Diederik Aerts [*]

D. Aerts in 1982 proposed a **macroscopic experiment** that violates the CHSH Bell Inequality maximally.

Two vessels V1 and V2 with a capacity of 8 liters each, linked through a tube with a capacity of 16 liters (at most the system holds 32 liters). The vessel Vref used to siphon water from the V1 and V2 basins.

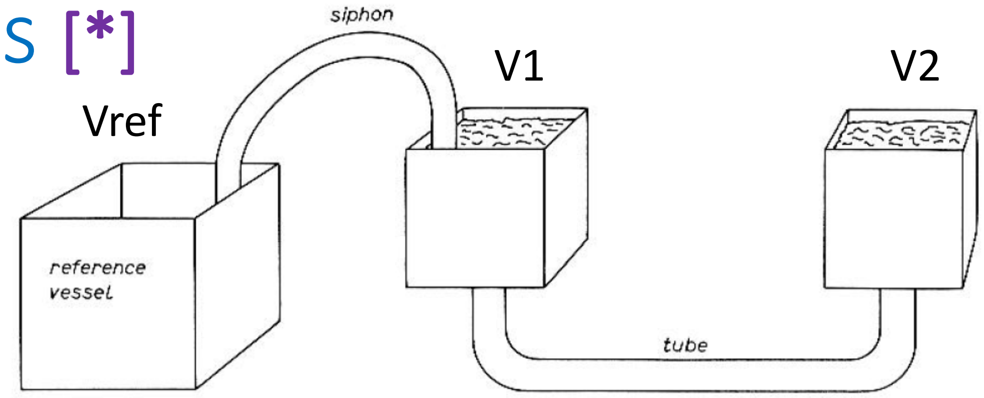


Fig. 1.

[*] D. Aerts, **Example of a macroscopical situation that violates Bell inequalities**, Lett. Nuovo Cimento **34**, 107 (1982)

4 experiments :

Experiment α: answer to: *is Vref > 10 L ?*

Experiment β: answer to: *are V1 or V2 > 6L ?*

Experiment γ: answer to: *is the water drinkable ?*

Experiment δ: answer to: *is the water transparent ?*

outcomes: $+1$ if the answer is **YES** and -1 if the answer is **NO**

Results of correlated experiments:

$X_{\alpha,\beta} = -1$, $X_{\alpha,\gamma} = +1$, $X_{\delta,\beta} = +1$ and $X_{\delta,\gamma} = +1$.

Taking the sum for the CHSH Bell parameter:

$$S = |X_{\alpha,\beta} - X_{\alpha,\gamma}| + |X_{\delta,\beta} + X_{\delta,\gamma}| = 4$$

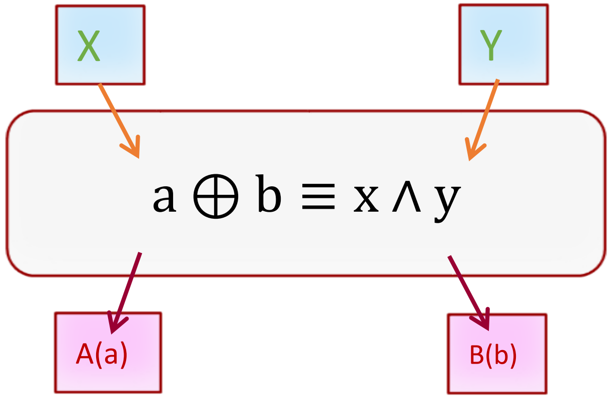
Bell's inequality is therefore maximally violated!

**In 2012 we undertook the experiment at Supélec using 2 flower pots and a 32 m water tube.
students: Vincent DUMOULIN & Yves SOURRILLE**

the logic of a PR Box

The well known nonlocal PR box [*] correlates outputs (a, b) to inputs (x, y) in a two-party correlation by means of a logical constraint equation:

$$a \oplus b \equiv x \wedge y$$



This box violates the CHSH Bell Inequality (BI) maximally
The measurement outcomes (A, B) , Alice and Bob, give the values ± 1.

We define the joint mean value for the possible outcomes of the box as a function of the marginal probabilities :

$$C_{x,y} = \sum_{a,b} P(a, b|a \oplus b \equiv x \wedge y) \cdot A(a) \cdot B(b)$$

where $A(a) = 1 - 2a = (-1)^{a}$; $B(b) = 1 - 2b = (-1)^{b}$

The Bell parameter considering the four input possibilities is:

$$S \;\; = C_{00} + C_{01} + C_{10} - C_{11} = 4$$

x	0	0	1	1
y	0	1	0	1
x ∧ y	0	0	0	1
C_{xy}	+1	+1	+1	-1
a ⊕ b	0	1	1	0
a , A	0 +1	0 +1	1 -1	1 -1
b , B	0 +1	1 -1	0 +1	1 -1

$$C_{xy} = P(0{,}0|x, y)^{\cdot}(+1)^{\cdot}(+1) + P(0{,}1|x, y)^{\cdot}(+1)^{\cdot}(-1) + P(1{,}0|x, y)^{\cdot}(-1)^{\cdot}(+1) + P(1{,}1|x, y)^{\cdot}(-1)^{\cdot}(-1)$$

$$C_{00} = C_{01} = C_{10} = \frac{1}{2} + 0 + 0 + \frac{1}{2} = +1$$

$$C_{11} = 0 - \frac{1}{2} - \frac{1}{2} + 0 = -1$$

[*] Popescu S., Rohrlich D. **Quantum nonlocality as an axiom**. Found Phys 24, 379–385 (1994)

analysing the PR Box Bell inequality by Eigenlogic

One uses the logical expression directly in an operator form using the following logical identity on the equivalence connective $\equiv$ leading to the Reed-Muller form:

$$\mathrm{a} \oplus \mathrm{b} \equiv \mathrm{x} \wedge \mathrm{y} \qquad \longleftrightarrow \qquad \overline{\mathrm{a} \oplus \mathrm{b} \oplus \mathrm{x} \wedge \mathrm{y}}$$

Using the involution properties: $(-1)^{\overline{\mathrm{a}\oplus\mathrm{b}\oplus\mathrm{x}\wedge\mathrm{y}}} = -(-1)^{\mathrm{a}\oplus\mathrm{b}\oplus\mathrm{x}\wedge\mathrm{y}} = -(-1)^{\mathrm{a}}(-1)^{\mathrm{b}}(-1)^{\mathrm{x}\wedge\mathrm{y}}$

One can then express the involution $\boldsymbol{G}_{\mathrm{PR}}$ operator's eigenvalues (truth values) by: $-(-1)^{a}(-1)^{b}(-1)^{xy}$

The corresponding Eigenlogic projective operator is: $\boldsymbol{F}_{\mathrm{PR}} = \frac{1}{2}(\mathbb{I} - \boldsymbol{G}_{\mathrm{PR}})$

The Bell parameter S is obtained by averaging the operator $\boldsymbol{G}_{\mathrm{PR}}$ on the possible situations given by the logical constraint $\mathrm{a} \oplus \mathrm{b} \equiv \mathrm{x} \wedge \mathrm{y}$ that is to the 8 cases out of 16 where the truth value of $\boldsymbol{F}_{\mathrm{PR}}$ is 1

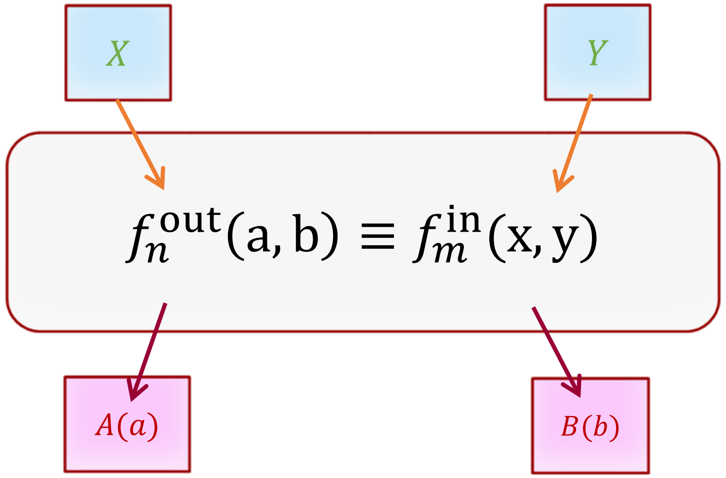
Considering all the possible cases for $C_{x,y}$ one gets the maximum Bell parameter:

using of the idempotent property: $\boldsymbol{F}_{\mathrm{PR}}{}^{2} = \boldsymbol{F}_{\mathrm{PR}}$

$$S = -\frac{8}{16}\mathrm{Tr}(\boldsymbol{F}_{\mathrm{PR}} \cdot \boldsymbol{G}_{\mathrm{PR}}) = -\frac{1}{2}\mathrm{Tr}\big(\boldsymbol{F}_{\mathrm{PR}}(\mathbb{I} - 2\boldsymbol{F}_{\mathrm{PR}})\big) = \frac{1}{2}\mathrm{Tr}(\boldsymbol{F}_{\mathrm{PR}}) = \frac{8}{2} = 4$$

Generalizing the PR Box BI for all logical bipartite constraints [*]

All combinations produce $16 \times 16 = 256$ logical constraint boxes.



The PR box corresponds to : $\mathrm{a} \oplus \mathrm{b} \equiv \mathrm{x} \wedge \mathrm{y} \quad (f_6(\mathrm{a},\mathrm{b}) \equiv f_8(\mathrm{x},\mathrm{y}))$

BI violation $|S| = \mathbf{4}$ for 16 no-signaling nonlocal boxes (**orange boxes**)

Other 32 boxes (**green boxes**) are signaling and violate BI with $|S| = \frac{10}{3} \approx \mathbf{3.33}$

is the case for: $\mathrm{a} \vee \mathrm{b} \equiv \mathrm{x} \wedge \mathrm{y} \quad (f_{14}(\mathrm{a},\mathrm{b}) \equiv f_8(\mathrm{x},\mathrm{y}))$

Logical function for *p, q* $f_n(p,q)$	Truth table for 2 inputs (p,q) (1,1) (1,0) (0,1) (0,0)	Logical operator	
f_0	(0 0 0 0)	contradiction :	False ; ⊥
f_1	(0 0 0 1)	not or :	NOR ; ¬P ∧ ¬Q
f_2	(0 0 1 0)	non-Implication :	P ⊇ Q
f_3	(0 0 1 1)	negation of p :	¬P
f_4	(0 1 0 0)	converse non-implication :	P ⊆ Q
f_5	(0 1 0 1)	negation of q :	¬Q
f_6	(0 1 1 0)	exclusive or :	XOR ; P ⊕ Q
f_7	(0 1 1 1)	not and :	NAND ; ¬P ∨ ¬Q
f_8	(1 0 0 0)	conjunction (and) :	AND ; P ∧ Q
f_9	(1 0 0 1)	equivalence :	XNOR ; P ≡ Q
f_{10}	(1 0 1 0)	right projection of q :	Q
f_{11}	(1 0 1 1)	converse implication	P ⊂ Q
f_{12}	(1 1 0 0)	left projection of p :	P
f_{13}	(1 1 0 1)	implication :	P ⊃ Q
f_{14}	(1 1 1 0)	disjunction (or) :	OR ; P ∨ Q
f_{15}	(1 1 1 1)	tautology :	True ; ⊤

OUTPUT / INPUT	f0	f1	f2	f4	f8	f3	f5	f12	f10	f6	f9	f7	f11	f13	f14	f15
f0																
f1										4	4	3,33	3,33	3,33	3,33	
f2										4	4	3,33	3,33	3,33	3,33	
f4										4	4	3,33	3,33	3,33	3,33	
f8										4	4	3,33	3,33	3,33	3,33	
f3																
f5																
f12																
f10																
f6																
f9																
f7		3,33	3,33	3,33	3,33					4	4					
f11		3,33	3,33	3,33	3,33					4	4					
f13		3,33	3,33	3,33	3,33					4	4					
f14		3,33	3,33	3,33	3,33					4	4					
f15																

[*] Z. Toffano, C. Chatard, K. Ding, A. Portalier, G. Vilde, **Logical Families of Nonlocal Boxes**, 6th Colloquium of the CNRS GDR 3322 on Q. Information, Foundations & Applications – IQFA (2015), ***Supélec student II year project in 2015***

42

Qubits and quantum circuits

Eigenlogic and 2-qubit quantum gates

Eigenlogic makes a correspondence between **quantum control logic** (David Deutsch's quantum logical gate paradigm) and ordinary propositional logic.
It is known that the 2-quibit **control-phase gate** $\boldsymbol{C_Z}$ in association with 1-quibit gates is a universal gate set.

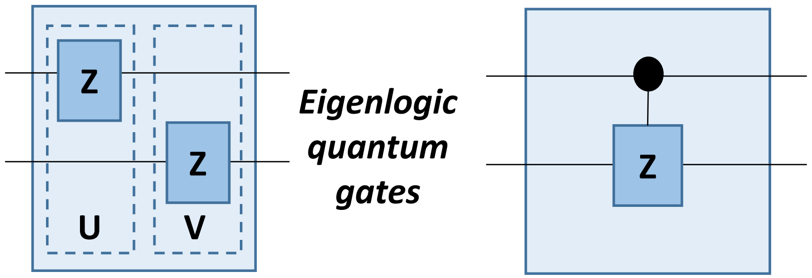
In Eigenlogic the $\boldsymbol{C_Z}$ gate corresponds to the AND involution gate $\boldsymbol{G}_\wedge$: $\boldsymbol{C_Z} = \boldsymbol{G}_\wedge = diag(1,1,1,-1)$

The well-known control-not **CNOT** *gate* $\boldsymbol{C}$ can be expressed using the Pauli matrices $\boldsymbol{\sigma_z} = \boldsymbol{Z}$ and $\boldsymbol{\sigma_x} = \boldsymbol{X}$

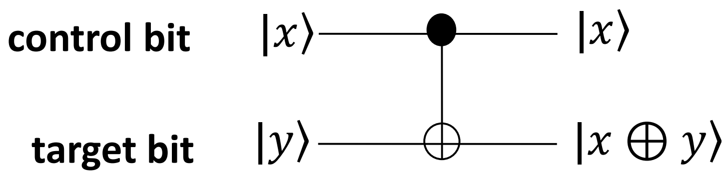
using the **Eigenlogic involution conjunction operator** (in the alphabet $\{+1,-1\}$) one has directly:

$$\boldsymbol{C} = (-1)^{\boldsymbol{\Pi} \otimes \boldsymbol{\Pi}_-} = e^{i\pi \boldsymbol{\Pi} \otimes \boldsymbol{\Pi}_-} = \mathbb{I} - 2(\boldsymbol{\Pi} \otimes \boldsymbol{\Pi}_-) = \frac{1}{2}(\mathbb{I} + \boldsymbol{Z} \otimes \mathbb{I} + \mathbb{I} \otimes \boldsymbol{X} - \boldsymbol{Z} \otimes \boldsymbol{X}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\boldsymbol{G}_\oplus = \boldsymbol{U}.\boldsymbol{V}$ $\quad\quad$ $\boldsymbol{C_Z} = \boldsymbol{G}_\wedge = ½(\boldsymbol{I} + \boldsymbol{U} + \boldsymbol{V} - \boldsymbol{U}.\boldsymbol{V})$



$\mathbf{CNOT} = \boldsymbol{C} = ½(\boldsymbol{I} + \boldsymbol{U} + \boldsymbol{V}_x - \boldsymbol{U}.\boldsymbol{V}_x)$



building the 3-qubit Toffoli universal quantum gate

The 3-qubit Toffoli (double-CNOT) gate $\boldsymbol{TO}$ is a **universal** reversible logic quantum gate, directly in Eigenlogic form :

$$\boldsymbol{TO} = (-1)^{\boldsymbol{\Pi}\otimes\boldsymbol{\Pi}\otimes\boldsymbol{\Pi}_-} = \mathbb{I} - 2(\boldsymbol{\Pi}\otimes\boldsymbol{\Pi}\otimes\boldsymbol{\Pi}_-)$$
$$= \frac{1}{4}(3\,\mathbb{I} + \boldsymbol{Z}_2 + \boldsymbol{Z}_1 + \boldsymbol{X}_0 - \boldsymbol{Z}_2\cdot\,\boldsymbol{Z}_1 - \boldsymbol{Z}_2\cdot\boldsymbol{X}_0 - \boldsymbol{Z}_1\cdot\boldsymbol{X}_0 + \boldsymbol{Z}_2\cdot\boldsymbol{Z}_1\cdot\boldsymbol{X}_0)$$

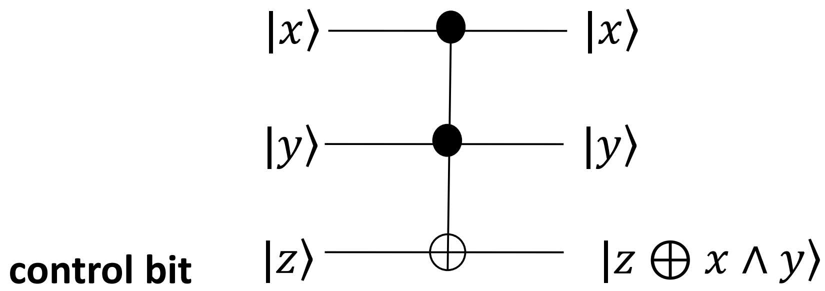
Can be put in exponential form using the Householder transform

$$\boldsymbol{TO} = e^{+i\frac{\pi}{8}}e^{-i\frac{\pi}{8}\boldsymbol{Z}_1}e^{-i\frac{\pi}{8}\boldsymbol{Z}_2}e^{-i\frac{\pi}{8}\boldsymbol{X}_0}e^{i\frac{\pi}{8}\boldsymbol{Z}_2\cdot\boldsymbol{Z}_1}e^{i\frac{\pi}{8}\boldsymbol{Z}_2\cdot\boldsymbol{X}_0}e^{i\frac{\pi}{8}\cdot\boldsymbol{Z}_1\cdot\boldsymbol{X}_0}e^{-i\frac{\pi}{8}\boldsymbol{Z}_2\cdot\boldsymbol{Z}_1\cdot\boldsymbol{X}_0}$$

Alternative method using a **T gate** as the Eigenlogic seed operator: $\boldsymbol{T} = \boldsymbol{Z}^{\frac{1}{4}} = e^{i\frac{\pi}{8}}e^{-i\frac{\pi}{8}\boldsymbol{Z}} = \mathrm{diag}\left(1, e^{i\frac{\pi}{4}}\right)$

using a **Reed-Muller form** for the CCZ gate [*] : $\boldsymbol{C}_{\boldsymbol{CZ}} = \boldsymbol{T}_0\cdot\boldsymbol{T}_1\cdot\boldsymbol{T}_2\cdot(\boldsymbol{T}_{x\oplus y})^\dagger\cdot(\boldsymbol{T}_{x\oplus z})^\dagger\cdot(\boldsymbol{T}_{y\oplus z})^\dagger\cdot(\boldsymbol{T}_{x\oplus y\oplus z})$

by the **Hadamard gate** extension one has again the Toffoli gate : $\boldsymbol{TO} = \boldsymbol{H}_0\cdot\boldsymbol{C}_{\boldsymbol{CZ}}\cdot\boldsymbol{H}_0$



$$\boldsymbol{TO} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

[*] Selinger, P., **Quantum circuits of T-depth one**, Phys. Rev. A, 87, 252–259, (2013)

logical universality and quantum entanglement

In binary propositional logic the following set of **8 connectives** when combined with negation (NOT) constitutes

a **universal gate set** : AND, OR, NOR, NAND, ⇒ , ⇏, ⇐ , ⇍

<u>one observes that they possess an **odd number of True and False truth values**</u>

the other 8 connectives are not universal: P, Q, ¬P, ¬Q, ≡ , XOR, F ,T

<u>one observes that they possess an **even number of True and False truth values**</u>

For involution logical operators $\boldsymbol{G}$ with eigenvalues $\{+1, -1\}$

the universal logical gates correspond to 8 operators with an <u>odd number of eigenvalues</u> $+1$ and -1

<u>therefore these are all entangling gates</u>

the 8 other logical operators with <u>even number of eigenvalues</u> are separable (not entangled) and not universal.

This states clearly the <u>correspondence between logical universality and entanglement</u>.

P	Q	F	NOR	P ⇍ Q	¬P	P ⇏ Q	¬Q	XOR	NAND	AND	P ≡ Q	Q	P ⇒ Q	P	Q ⇒ P	OR	T
+	+	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
+	−	+	+	−	−	−	+	−	−	+	+	−	−	+	+	−	−
−	+	+	+	+	+	+	−	−	−	+	+	+	+	−	−	−	−
−	−	+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−

Deutsch algorithm [*]

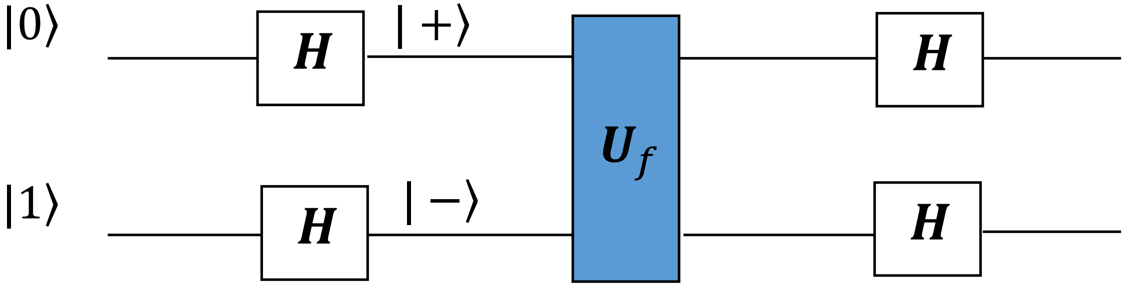
The Deutsch algorithm is <u>one of the first quantum algorithms more efficient than its classical counterpart</u>.

> The answer to the question: is the <u>logical function $f(x)$</u> constant or balanced? can be performed by a quantum computer in one step.
> (the classic treatment requires two steps.)

x	$f_{00}(x)$	$f_{01}(x)$	$f_{10}(x)$	$f_{11}(x)$
0	0	0	1	1
1	0	1	0	1
	constant	balanced	balanced	constant

[*] D. Deutsch, **Quantum theory, the Church-Turing principle and the universal quantum computer**, Proc. R. Soc. A, 400, 97–117, (1985)

The algorithm measurement is made on the upper qubit of the following circuit



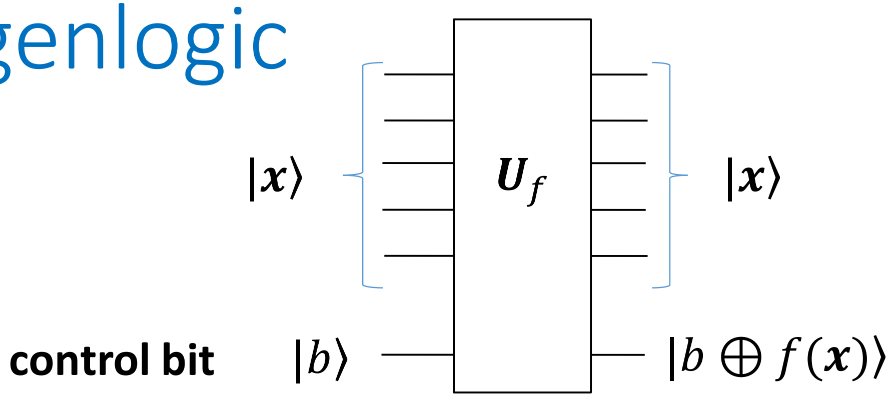
$$= (-1)^{f(0)}\,|f(0)\oplus f(1)\rangle|1\rangle$$

$$= (-1)^{f(0)}|0\rangle|1\rangle \text{ if constant}$$

$$= (-1)^{f(0)}|1\rangle|1\rangle \text{ if balanced}$$

$$U_f\!: |x\rangle|y\rangle \rightarrow |x\rangle|y\oplus f(x)\rangle$$

expressing the quantum oracle circuit in Eigenlogic



A Boolean logical function f is represented by a **quantum oracle** $\boldsymbol{U_f}$:

<u>Particular cases are</u>:
the 2-qubit CNOT gate $\boldsymbol{C}$ (f is the NOT function)
the 3-qubit Toffoli gate $\boldsymbol{TO}$ (f is the AND function)

The logical function f is represented by the projective Eignelogic operator $\boldsymbol{F}$

The control bit corresponds to the seed projection operator in the $|x\rangle$ basis: $\boldsymbol{\Pi}_- = |-\rangle\langle -| = \frac{1}{2}(\mathbb{I} - \boldsymbol{X})$

<u>The oracle is then simply expressed in Eigenlogic as</u> : $\boldsymbol{U_f} = (-1)^{\boldsymbol{F}\otimes\boldsymbol{\Pi}_-} = e^{i\pi\boldsymbol{F}\otimes\boldsymbol{\Pi}_-} = \mathbb{I} - 2\boldsymbol{F}\otimes\boldsymbol{\Pi}_-$
(similar approach by **S. Hadfield** in **[*]**)

In the case of a one bit Boolean function $f(x)$: $\boldsymbol{U_f} = \boldsymbol{\Pi}_0 \otimes \boldsymbol{X}^{f(0)} + \boldsymbol{\Pi}_1 \otimes \boldsymbol{X}^{f(1)}$

the **Deutsch algorithm** result is obtained by applying the oracle on : $|+\rangle|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

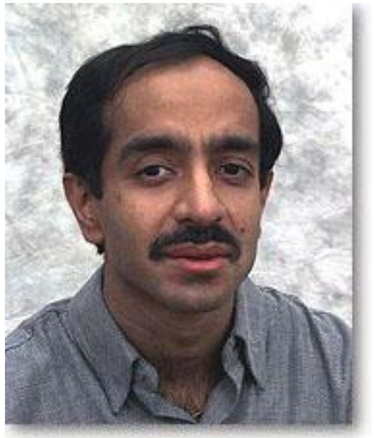
$$\boldsymbol{U_f}|+\rangle|-\rangle = \frac{1}{\sqrt{2}}|0\rangle(-1)^{f(0)}|-\rangle + \frac{1}{\sqrt{2}}|1\rangle(-1)^{f(1)}|-\rangle$$

for constant $f(0) = f(1)$ $\quad \boldsymbol{U_f}|+\rangle|-\rangle = \pm|+\rangle|-\rangle$ $\quad$ and for balanced $f(0) \neq f(1)$ $\quad \boldsymbol{U_f}|+\rangle|-\rangle = \pm|-\rangle|-\rangle$

[*] S. Hadfield, **On the Representation of Boolean and Real Functions as Hamiltonians for Quantum Computing**.
to appear in ACM Transactions on Quantum Computing, arXiv:1804.09130

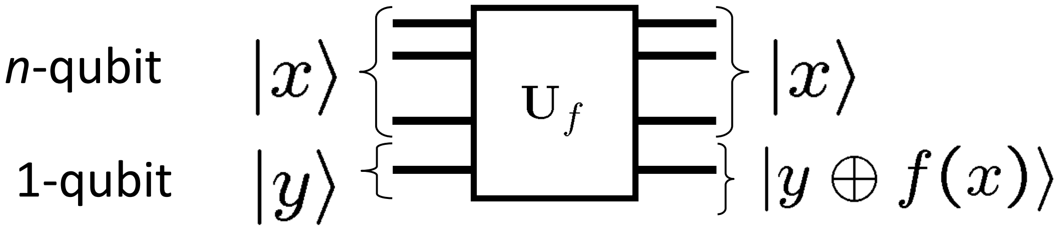
Grover's search algorithm [*]

Problem: finding the value x_0 in a large database with the fewest queries possible.



We consider a "black box" (oracle $\mathbf{U}_f$)
Having for the value x_0 the property :
$f(x_0) = 1$ and $f(x) = 0$ for $x \neq x_0$

[*] L. K. Grover. **A fast quantum mechanical algorithm for database search**, Proceedings, 28th Annual ACM Symp. on the Theory of Computing, p. 212, (1996)



Classical query: complexity: $\boldsymbol{O}(\mathbf{exp}(\boldsymbol{n}))$

Grover quantum query: complexity: $\boldsymbol{O}(\mathbf{exp}(\sqrt{\boldsymbol{n}}))$

Grover algorithm and first-order-logic

The **Grover search algorithm** looks for an element a (here $|a_2 a_1 a_0\rangle = |111\rangle$) satisfying the property P (**oracle**) and becomes the predicate proposition in **first-order-logic** using the **existential logical quantifier ∃** :

$$\exists a\ P(a)$$

The **Grover amplification gate** corresponds to an Eigenlogic negated disjunction operator NOR in the X system.

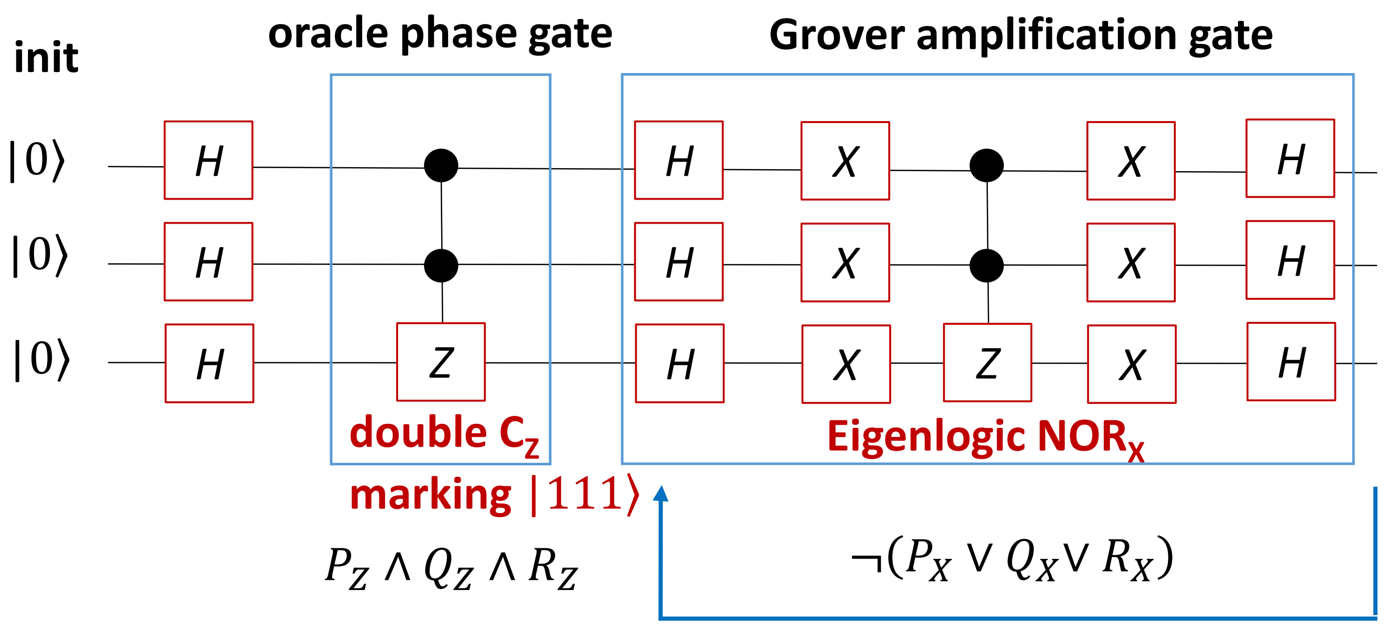
The **phase oracle** is a **double control Z gate** (Eigenlogic 3-input AND: $(-1)^{\boldsymbol{\Pi}\otimes\boldsymbol{\Pi}\otimes\boldsymbol{\Pi}}$).

$\exists a\ P(a)$ decomposes, using ***Skolemization*** methods into a succession of "disjunction ∨" connectives.

$$\exists a \iff a_1 \vee a_2 \vee \cdots \vee a_N$$

$$\forall a \iff a_1 \wedge a_2 \wedge \cdots \wedge a_N$$

A justification can be found in **Herbrand's theorem** that provides a constructive characterization of derivability in first-order predicate logic by means of propositional (sentential) logic.



Quantum robots

quantum robots: Paul Benioff

Paul Benioff was the first to propose the idea of a quantum Turing machine in 1980 [1].

Benioff also gave the theoretical principle of a **quantum robot** in 1988 [2] as a first step towards a quantum mechanical description of systems that are aware of their environment and make decisions.

Currently a “Quantum Robotics” group has been created and a book has been published in 2017 [3].

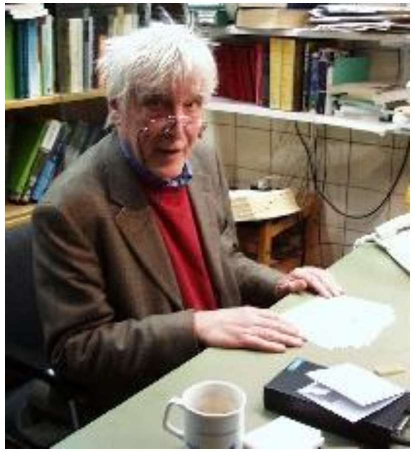
“Quantum Robotics is an emerging engineering and scientific research discipline that explores the application of quantum mechanics, quantum computing, quantum algorithms, and related fields to robotics. These developments are expected to impact the technical capability for robots to sense, plan, learn, and act in a dynamic environment.”

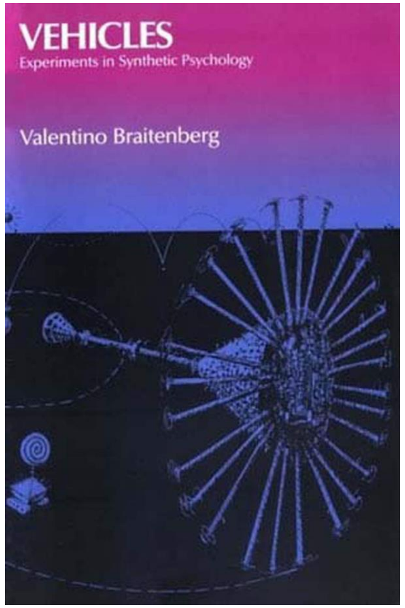
[1] Benioff, P., **The computer as a physical system: A microscopic quantum mechanical Hamiltonian model of computers as represented by Turing machines**, J. of Statistical Phys., Vol. 22 (1980)

[2] Benioff, P., **Quantum Robots and Environments**, Physical Review A, Vol. 58, No.2, pp. 893–904 (1988)

[3] Tandon, P., Lam, S., Shih, B., Mehta, T., Mitev, A., Ong, Z., **Quantum Robotics - A Primer on Current Science and Future Perspectives**, Morgan & Claypool. (2017)

Braitenberg Vehicles (BV)





Valentino Braitenberg was a cyberneticist at the University of Tübingen.

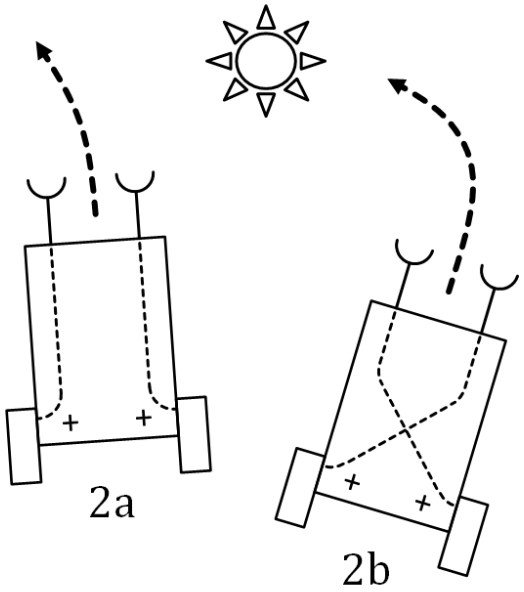
In his book *Vehicles* **[1]** he proposed various thought experiments using simple machines consisting in sensors, motors and wheels.

Sensors detect light produced by surrounding sources.
The sensors can be connected in different configurations of combinations to the wheels. Simple changes in configuration can lead to complex and surprising results in the agent behavior.

We designed *quantum-like* Braitenberg vehicles **[2]**.

The control is based on Fuzzy Eigenlogic.

The goal is to test the multiple combinations of logical gates used in the control of Braitenberg vehicles by analyzing their complex behavior.

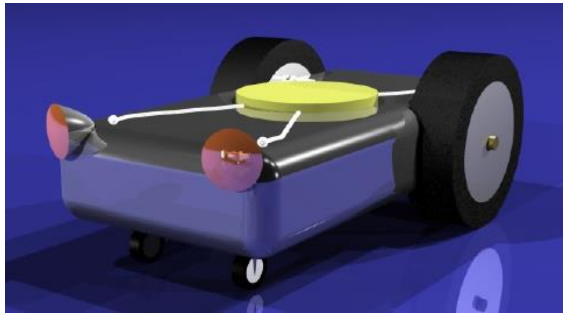


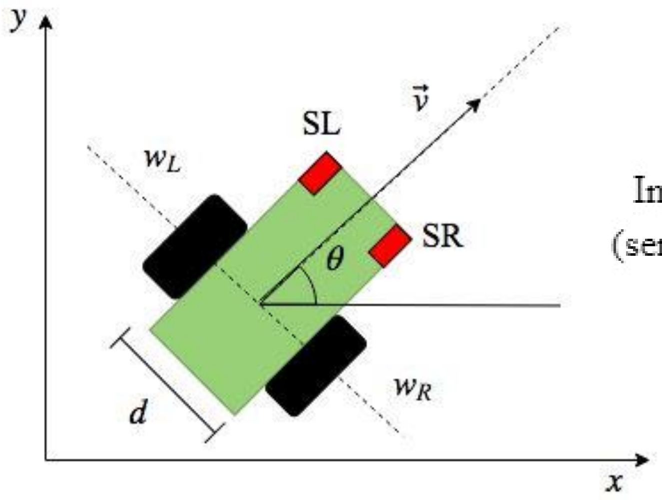
[1] Braitenberg, V., **Vehicles - Experiments in Synthetic Psychology**. MIT Press; Cambridge USA. (1986)

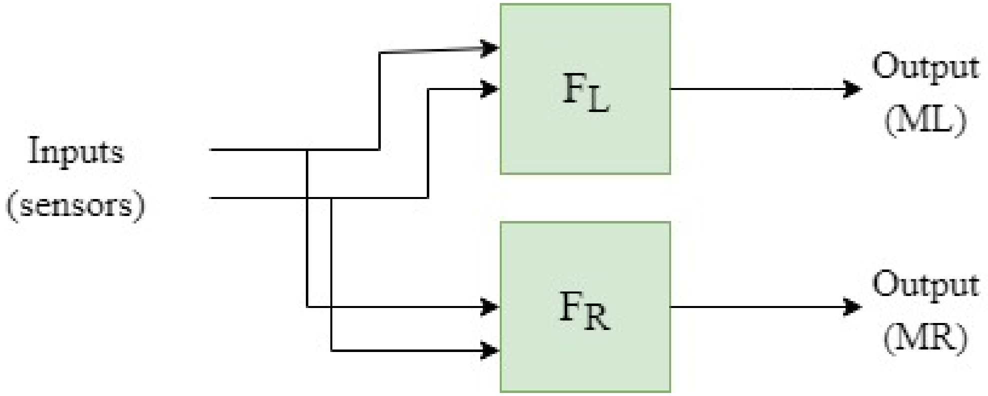
[2] Z. Toffano, F. Dubois, **Quantum eigenlogic observables applied to the study of fuzzy behaviour of Braitenberg vehicle quantum robots**, Kybernetes, (2019)

vehicle input-output structure [*]

The computational block is composed of logic operators (matrices) designed with the quantum-like Eigenlogic method. The control of the two-wheel motors (*ML*, *MR*), responds to signals from the two light sensors (*SL*, *SR*).







the input state vector representing the light intensity on *SL* and *SR* is: $|\psi\rangle = |\phi_{SL}\rangle \otimes |\phi_{SR}\rangle$

The fuzzy quantities for left and right wheel (*WL*, *WR*) control, are the mean values of the logical operators $\boldsymbol{F_L}$ and $\boldsymbol{F_R}$ on the input compound state $|\psi\rangle$:

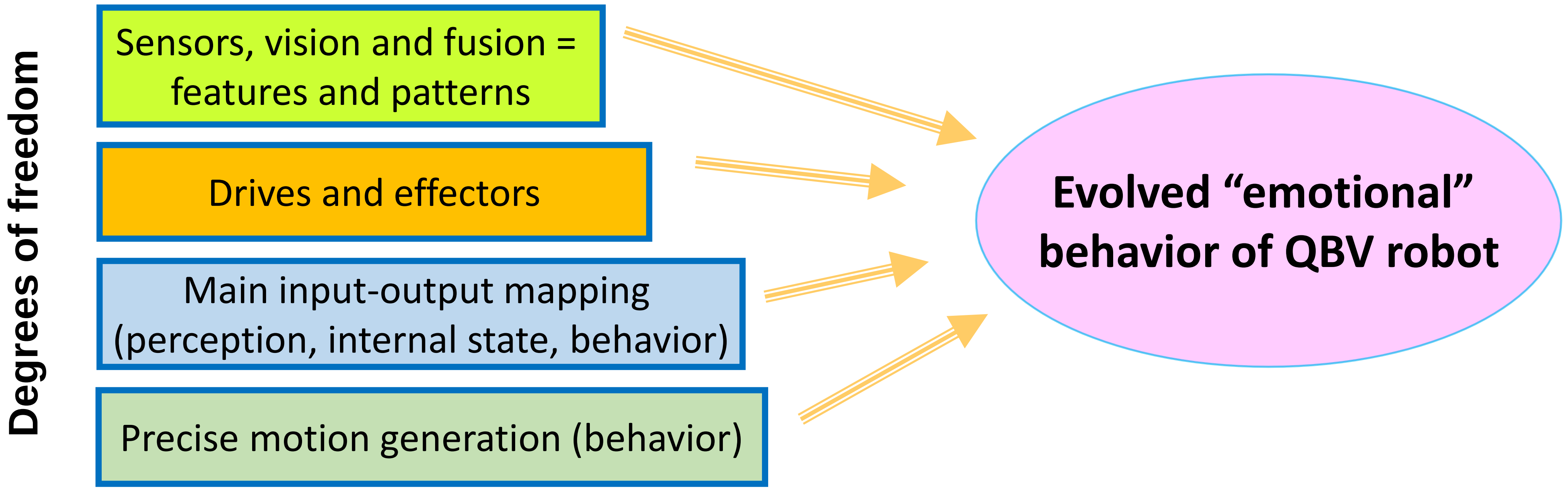
$$\mu_L = \langle\psi|\boldsymbol{F_L}|\psi\rangle \quad , \quad \mu_R = \langle\psi|\boldsymbol{F_R}|\psi\rangle$$

[*] Cunha R.A.F., Sharma N., Toffano Z., Dubois F. **Fuzzy Logic Behavior of Quantum-Controlled Braitenberg Vehicle Agents**. In: Coecke B., Lambert-Mogiliansky A. (eds) Quantum Interaction. QI 2018. Lecture Notes in Computer Science, vol 11690. Springer, Cham. (2019)

emotion as emergent, evolvable behavior

Emotion is an emergent behavior that arises from sensors, drives, effectors and logic.

This may look like human or animal behavior but also as an entirely new “other world” behavior.



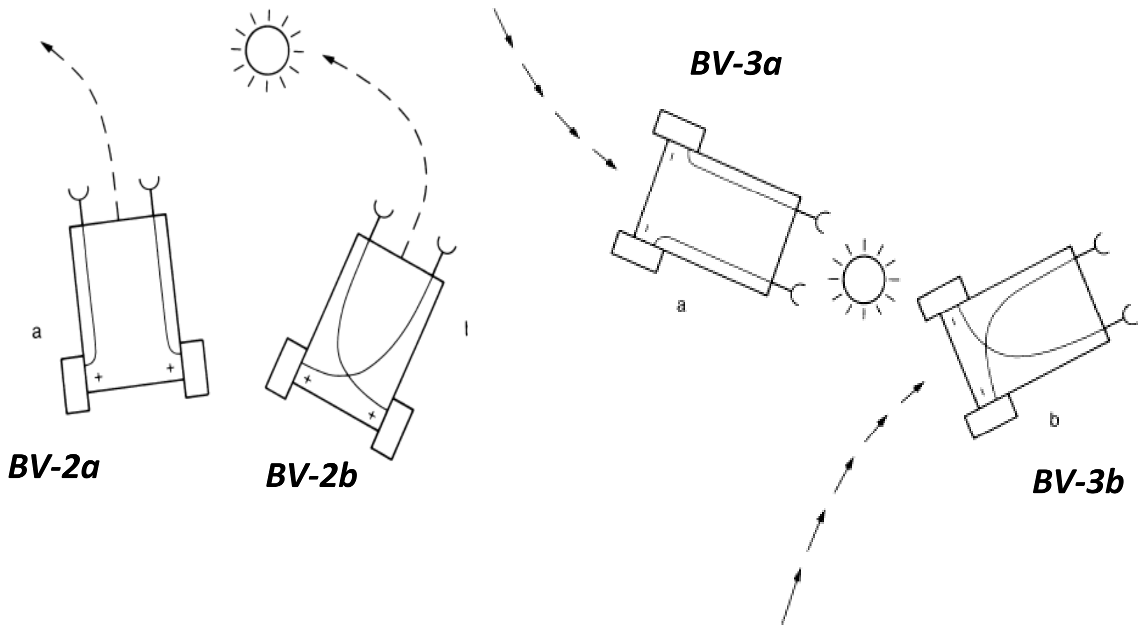
BV classical emotions

Fear *(BV-2a)* ($\boldsymbol{F}_L = \boldsymbol{A}$ and $\boldsymbol{F}_R = \boldsymbol{B}$):
turns away from the light if one sensor is activated more than the other. If both are equal, the light source is "attacked".

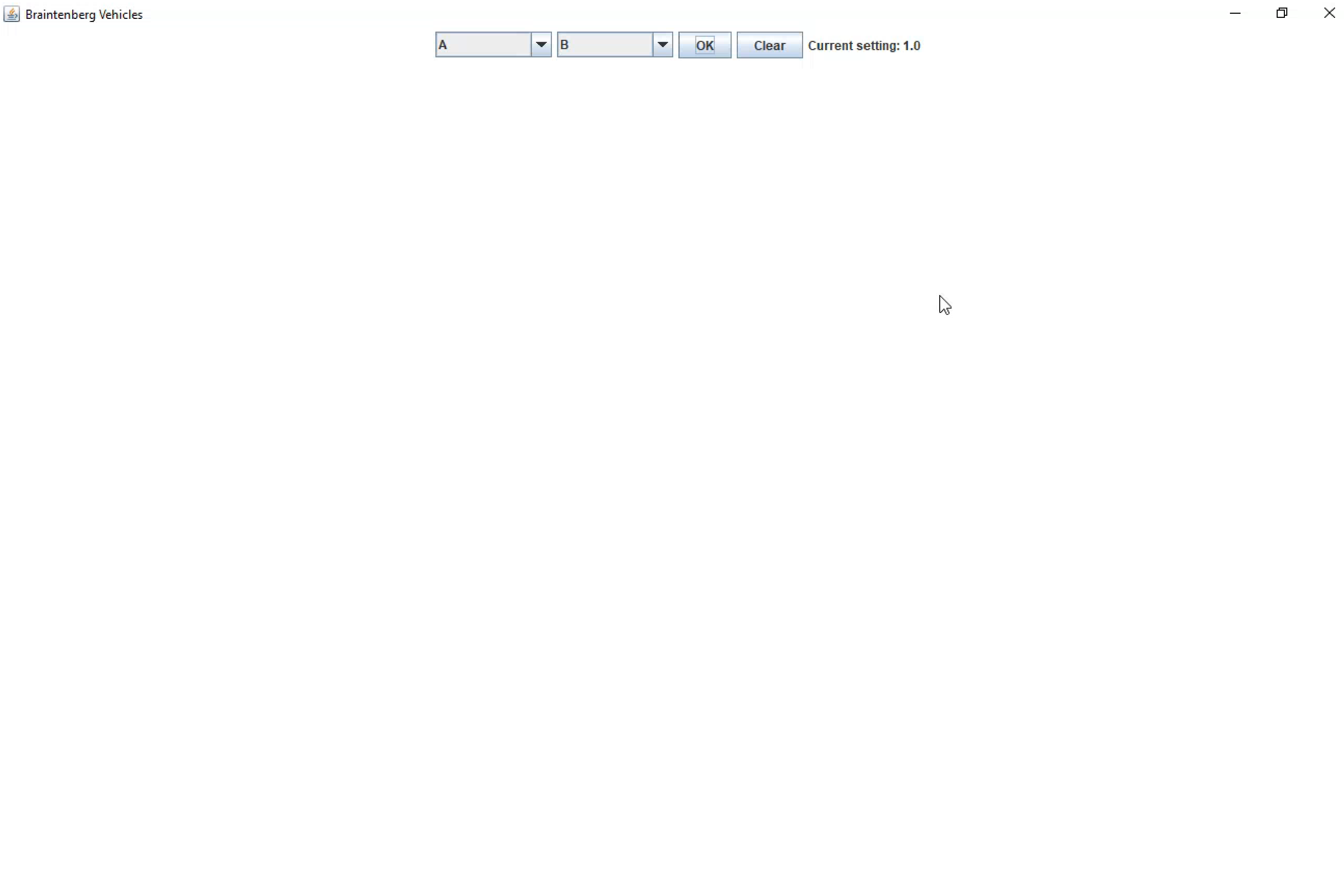
Aggression *(BV-2b)* ($\boldsymbol{F}_L = \boldsymbol{B}$ and $\boldsymbol{F}_R = \boldsymbol{A}$):
when the light source is placed near either sensor, the vehicle will face it and go towards it.

Love *(BV-3a)* ($\boldsymbol{F}_L = \mathbb{I} - \boldsymbol{A}$ and $\boldsymbol{F}_R = \mathbb{I} - \boldsymbol{B}$):
will go until it finds a light source, then slows to a stop. If one side sees light, the vehicle turns in the direction of the light.

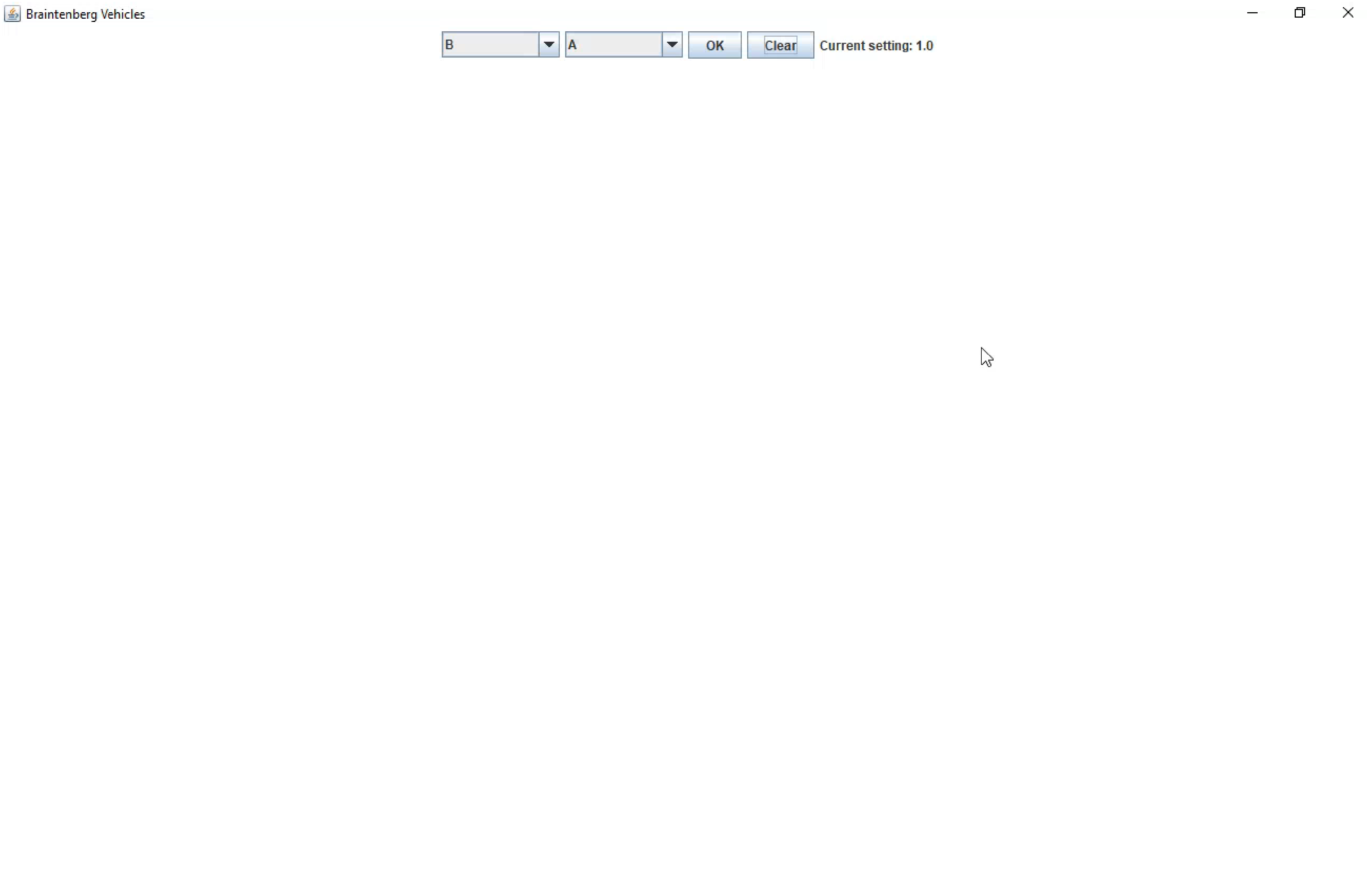
Exploration *(BV-3b)* ($\boldsymbol{F}_L = \mathbb{I} - \boldsymbol{B}$ and $\boldsymbol{F}_R = \mathbb{I} - \boldsymbol{A}$):
will go to the nearby light source, keeps an eye open and will sail to other stronger sources, given the chance.



Fear



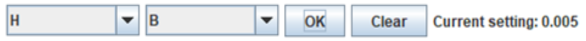
Aggression



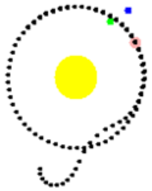
BV quantum-like emotions: worship, doubt

<u>Worship</u>: ($\boldsymbol{F}_L = \frac{1}{2}(1 - \boldsymbol{H} \otimes \boldsymbol{H})$ and $\boldsymbol{F}_R = \boldsymbol{B}$):
one of the control logical operator is the projector version of the double-Hadamard qubit gate $\boldsymbol{H}$.

The vehicle keeps rotating around its own center in the absence of light. In the presence of light, it goes towards the source and starts to rotate around the source (or multiple sources when they are close together).



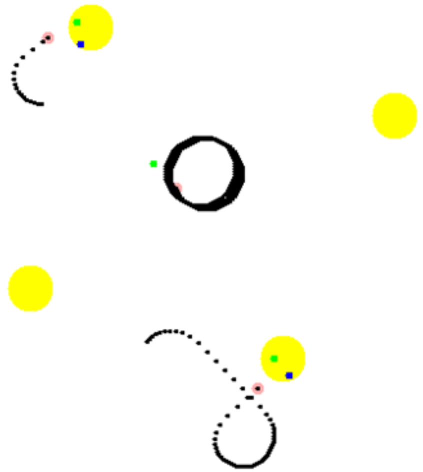
$\lvert x_f \rangle$	μ_L	μ_R	Behavior
$\lvert 00 \rangle$	0.25	0	turns to the right slowly
$\lvert 01 \rangle$	0.75	1	turns to the left slowly
$\lvert 10 \rangle$	0.75	0	turns to the right
$\lvert 11 \rangle$	0.25	1	turns to the left



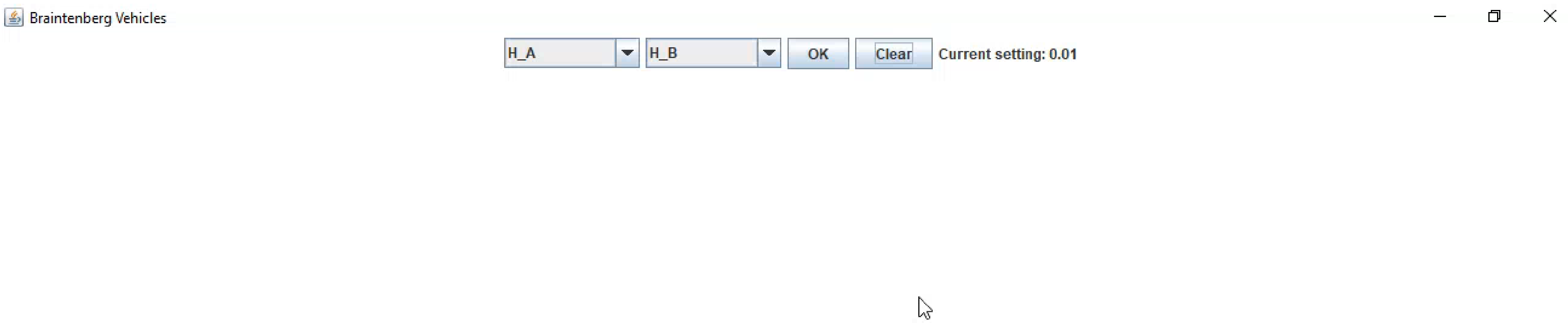
<u>Doubt</u>: ($\boldsymbol{F}_L = \boldsymbol{B} \not\Rightarrow \boldsymbol{A}$ and $\boldsymbol{F}_R = \boldsymbol{F}_{XOR} = \frac{1}{2}(1 - \boldsymbol{Z} \otimes \boldsymbol{Z})$):
here, one of the control logical operator is *XOR* (projector version of the double- $\boldsymbol{Z}$ qubit gate).

This operator provides a property that makes the vehicle turn around in circles, regardless of the presence or absence of stimuli.

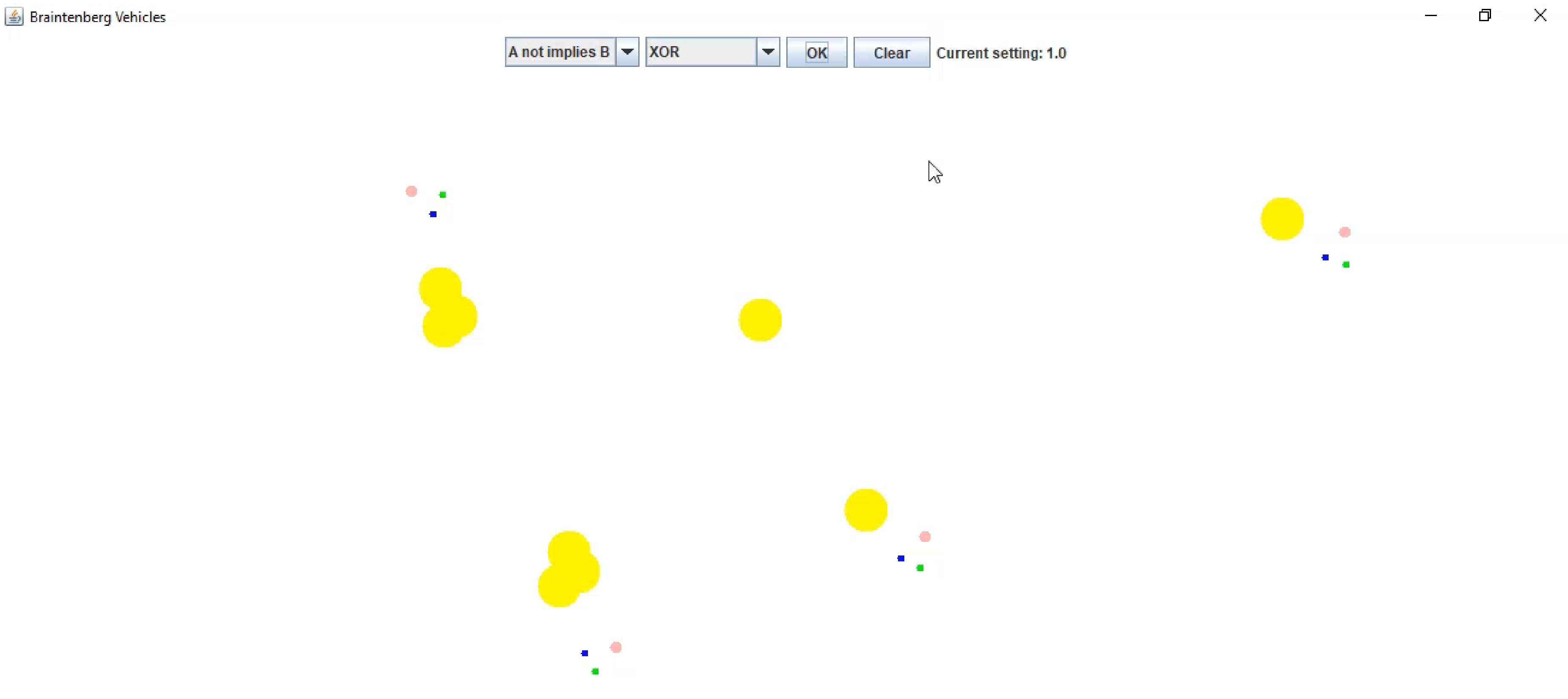
$\lvert x_f \rangle$	μ_L	μ_R	Behavior
$\lvert 00 \rangle$	0	0	no movement
$\lvert 01 \rangle$	1	1	goes straight
$\lvert 10 \rangle$	0	1	turns to the left
$\lvert 11 \rangle$	0	0	no movement



Worship



Doubt


Braintenberg Vehicles
A not implies B
XOR
OK
Clear
Current setting: 1.0

quantum wheel of emotions

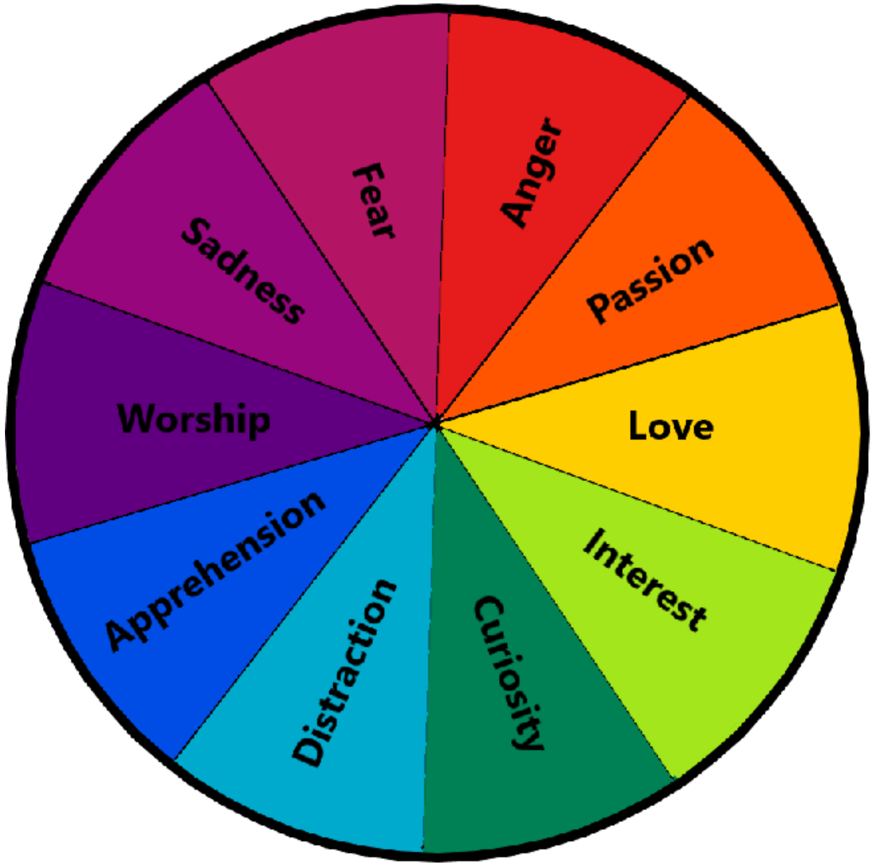
The concept of “wheel of emotions” introduced by R. Plutchik **[*]** allows a continuous set of emotional states.

A small perturbation in the angle of the input state will correspond to small changes in the vehicle’s behavior.

The emotions presented on the wheel shown are an example corresponding to the following $L \mid R$ logical control operators:

- (Anger-Aggressive) $B \mid A$
- (Passion) $B \nRightarrow A \mid A \nRightarrow B$
- (Love) $\bar{A} \mid \bar{B}$
- (Interest-Explore) $\bar{B} \mid \bar{A}$
- (Curiosity) $A \nRightarrow B \mid B \nRightarrow A$
- (Distraction) $A \nRightarrow B \mid XOR$
- (Apprehension) $B \nRightarrow A \mid XOR$
- (Worship) $\boldsymbol{H} \otimes \boldsymbol{H} \mid B$
- (Sadness) $CNOT \mid CNOT$
- (Fear) $A \mid B$

[*] Plutchik, R., 2001, **The Nature of Emotions**, Amercian Scientist, July-August, Vol. 89, N° 4, pp. 334-350



Thanks to people related with this research

François Dubois, CNAM & U-Paris-Sud , France

Francesco Galofaro, U. Torino, Italy

Benoît Valiron, CentraleSupélec, France

Bich-Lien Doan, CentraleSupélec, France

Alberto Ottoloenghi, UCL, UK

Andrei Khrennikov, Linnaeus U., Sweden

Valentino Braitenberg, U. Tübingen, Germany

Gianfranco Basti, Lateran U., Vatican

Jean-Yves Béziau, U. Rio de Janeiro, Brazil

interesting reads